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GEODESIC FLOW ON THE NORMAL CONGRUENCE OF A MINIMAL SURFACE

BRENDAN GUILFOYLE AND WILHELM KLINGENBERG

ABSTRACT. We study the geodesic flow on the normal line congruence of a minimal surface in \mathbb{R}^3 induced by the neutral Kähler metric on the space of oriented lines. The metric is lorentz with isolated degenerate points and the flow is shown to be completely integrable. In addition, we give a new holomorphic description of minimal surfaces in \mathbb{R}^3 and relate it to the classical Weierstrass representation.

1. INTRODUCTION

In a recent paper [4] a neutral Kähler metric was introduced on the space \mathbb{L} of oriented affine lines in \mathbb{R}^3 . This metric is natural in the sense that it is invariant under the action induced on \mathbb{L} by the Euclidean action on \mathbb{R}^3 . Moreover, a surface in \mathbb{L} is lagrangian with respect to the associated symplectic structure iff there exist surfaces orthogonal to the associated 2-parameter family of oriented lines (or line congruence) in \mathbb{R}^3 .

In this paper we characterise the set of oriented normals to a minimal surface in \mathbb{R}^3 and study the geodesic flow on the line congruence induced by this neutral Kähler metric. Along the way, we give a new holomorphic description of minimal surfaces in \mathbb{R}^3 and relate it to the classical Weierstrass representation.

The induced metric on a lagrangian line congruence is either lorentz or degenerate. The null geodesics of the lorentz metric correspond to the principal foliation on the orthogonal surface and the degeneracy occurs precisely at umbilic points. We show that on the normal congruence of a minimal surface the geodesic flow is completely integrable and find the first integrals. Recently the geodesic flow on certain non-lagrangian line congruences was investigated [5]. In that case, the metric was riemannian with degeneracies along a curve.

The picture that emerges is this: every minimal surface carries a completely integrable dynamical system [2] [7] [8]. This is generated by geodesic motion of a lorentz metric whose null geodesics are the lines of curvatures and whose sources are the isolated umbilic points of the minimal surface. To illustrate this we compute the geodesics explicitly for the case of pure harmonic minimal surfaces. These have a unique index $-N$ umbilic point (for $N > 0$) and we show that the scattering angle for non-null geodesics is $2\pi/(N + 2)$.

The next section describes the normal line congruence to a minimal surface - all background details on the geometry of the space of oriented affine lines in \mathbb{R}^3 can be found in [3] [4] and references therein. We relate the present work to the Weierstrass

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representation in Section 3. We then prove the result about the geodesic flow in Section 4, while we look at the case of pure harmonic minimal surfaces in the final section.

2. THE NORMAL LINE CONGRUENCE TO A MINIMAL SURFACE

Let \mathbb{L} be the space of oriented lines in \mathbb{R}^3 which we identify with the tangent bundle to the 2-sphere [6]. Let $\pi : \mathbb{L} \rightarrow \mathbb{P}^1$ be the canonical bundle and $(\mathbb{J}, \Omega, \mathbb{G})$ the neutral Kähler structure on \mathbb{L} [4].

A line congruence is a 2-parameter family of oriented lines in \mathbb{R}^3 , or equivalently, a surface $\Sigma \subset \mathbb{L}$. We are interested in characterising the line congruence formed by the oriented normal lines to a minimal surface S in \mathbb{R}^3 :

Theorem 1. *A lagrangian line congruence $\Sigma \subset \mathbb{L}$ is orthogonal to a minimal surface without flat points in \mathbb{R}^3 iff the congruence is the graph $\xi \mapsto (\xi, \eta = F(\xi, \bar{\xi}))$ of a local section of the canonical bundle with:*

$$\bar{\partial} \left(\frac{\partial \bar{F}}{(1 + \xi \bar{\xi})^2} \right) = 0, \quad (2.1)$$

where (ξ, η) are standard holomorphic coordinates on $\mathbb{L} - \pi^{-1}\{\text{south pole}\}$ and ∂ represents differentiation with respect to ξ .

Proof. Let S be a minimal surface without flat points and Σ be its normal line congruence. Since the line congruence is not flat, it can be given by the graph of a local section. In terms of the canonical coordinates $(\xi, \eta = F(\xi, \bar{\xi}))$ the spin coefficients of such a line congruence are [3]:

$$\rho = \frac{\psi}{\partial F \partial \bar{F} - \psi \bar{\psi}} \quad \sigma = -\frac{\partial \bar{F}}{\partial F \partial \bar{F} - \psi \bar{\psi}},$$

with

$$\psi = \partial F + r - \frac{2\xi \bar{F}}{1 + \xi \bar{\xi}}.$$

As this line congruence is orthogonal to a surface in \mathbb{R}^3 , ρ is real, and, as the mean curvature vanishes, $\rho = 0$ on S .

Now, the graph of a lagrangian section satisfies the following identity:

$$(1 + \xi \bar{\xi})^2 \bar{\partial} \left(\frac{\sigma_0}{(1 + \xi \bar{\xi})^2} \right) = -\partial \psi, \quad (2.2)$$

where we have introduced $\sigma_0 = -\partial \bar{F}$. This follows from the fact that partial derivatives commute: firstly the left-hand side is

$$(1 + \xi \bar{\xi})^2 \bar{\partial} \left(\frac{\sigma_0}{(1 + \xi \bar{\xi})^2} \right) = -\bar{\partial} \partial \bar{F} + \frac{2\xi \partial \bar{F}}{1 + \xi \bar{\xi}},$$

while the right-hand side is

$$-\partial \psi = -\partial \left(\bar{\partial} \bar{F} + r - \frac{2\xi \bar{F}}{1 + \xi \bar{\xi}} \right) = -\partial \bar{\partial} \bar{F} + \frac{2\xi \partial \bar{F}}{1 + \xi \bar{\xi}}.$$

Here we have used the lagrangian condition $\rho = \bar{\rho}$ and the equivalent local existence of a real function $r : \Sigma \rightarrow \mathbb{R}$ such that

$$\bar{\partial}r = \frac{2F}{(1 + \xi\bar{\xi})^2}. \quad (2.3)$$

Thus, since $\rho = 0$, we have $\psi = 0$ and according to the identity (2.2), the normal congruence to a minimal surface must satisfy the holomorphic condition (2.1).

Conversely, suppose (2.1) holds for a lagrangian line congruence Σ which is given by the graph of a local section. Then, by the identity (2.2) $\psi = C$ for some real constant C . As the orthogonal surfaces move along the line congruence in \mathbb{R}^3 , ψ changes by $\psi \rightarrow \psi + \text{constant}$. Thus there exists a surface S for which $\psi = 0$, and therefore $\rho = 0$, i.e. there is a minimal surface orthogonal to Σ . \square

The previous theorem has two immediate consequences:

Corollary 1. *The normal congruence to a minimal surface is given by a local section F of the bundle $\pi : \mathbb{L} \rightarrow S^2$ with*

$$F = \sum_{n=0}^{\infty} 2\lambda_n \xi^{n+3} - \bar{\lambda}_n \bar{\xi}^{n+1} \left((n+2)(n+3) + 2(n+1)(n+3)\xi\bar{\xi} + (n+1)(n+2)\xi^2\bar{\xi}^2 \right),$$

for complex constants λ_n . The potential function $r : \Sigma \rightarrow \mathbb{R}$ satisfying (2.3) is:

$$r = -2 \sum_{n=0}^{\infty} \frac{(3+n+(1+n)\xi\bar{\xi})(\lambda\xi^{n+2} + \bar{\lambda}\bar{\xi}^{n+2})}{1 + \xi\bar{\xi}}.$$

Proof. Since the minimal surface condition is a holomorphic condition we can expand in a power series about a point:

$$\frac{\partial\bar{F}}{(1 + \xi\bar{\xi})^2} = \sum_{n=0}^{\infty} \alpha_n \xi^n.$$

This can be integrated term by term to

$$\bar{F} = \sum_{n=0}^{\infty} \beta_n \bar{\xi}^n + \alpha_n \xi^{n+1} \left(\frac{1}{n+1} + \frac{2}{n+2}\xi\bar{\xi} + \frac{1}{n+3}\xi^2\bar{\xi}^2 \right),$$

for complex constants β_n . Now we impose the lagrangian condition, that

$$(1 + \xi\bar{\xi})\bar{\partial}\bar{F} - 2\xi\bar{F} = \sum_{n=0}^{\infty} \beta_n \bar{\xi}^{n-1} (n + (n-2)\xi\bar{\xi}) - 2\alpha_n \xi^{n+2} \left(\frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)}\xi\bar{\xi} \right),$$

is real. This implies that $\beta_0 = \beta_1 = \beta_2 = 0$ and $(n+1)(n+2)(n+3)\beta_{n+3} = -2\bar{\alpha}_n$ for $n \geq 0$. Letting $\alpha_n = -(n+1)(n+2)(n+3)\lambda_n$ gives the stated result.

Finally it is easily checked that the expressions for r and F satisfy (2.3). \square

On a minimal surface flat points are also umbilic points (and vice versa). Such points are now shown to be isolated:

Corollary 2. *Umbilic points on minimal surfaces are isolated and the index of the principal foliation about an umbilic point on a minimal surface is less than or equal to zero.*

Proof. An umbilic point is a point where $\partial\bar{F} = 0$.

Moreover, the argument of $\partial\bar{F}$ gives the principal foliation of the surface [3]. Given that minimality implies the holomorphic condition (2.1), the zeros of $\partial\bar{F}$ are isolated and have index greater than or equal to zero. \square

3. THE WEIERSTRASS REPRESENTATION OF A MINIMAL SURFACE

The classical Weierstrass representation constructs a minimal surface from a holomorphic curve in \mathbb{L} [6]. The minimal surface in \mathbb{R}^3 determined by a local holomorphic section $\nu \mapsto (\nu, w(\nu))$ of the canonical bundle is given by

$$\begin{aligned} z &= \frac{1}{2}w'' - \frac{1}{2}\bar{\nu}^2\overline{w''} + \bar{\nu}w' - \bar{w} \\ t &= \frac{1}{2}\nu w'' - \frac{1}{2}w' + \frac{1}{2}\bar{\nu}\overline{w''} - \frac{1}{2}\overline{w'}, \end{aligned}$$

where a prime represents differentiation with respect to the holomorphic parameter ν and $z = x^1 + ix^2$, $t = x^3$ for Euclidean coordinates (x^1, x^2, x^3) . The relationship between this and our approach is as follows.

Proposition 1. *The normal congruence of the minimal surface, in terms of the canonical coordinates ξ and η , is*

$$\xi = -\bar{\nu} \quad \eta = \frac{1}{4}(1 + \xi\bar{\xi})^3 \frac{\partial^2}{\partial \bar{\xi}^2} \left(\frac{w}{1 + \xi\bar{\xi}} \right) - \frac{1}{2}\bar{w}.$$

Proof. We have that

$$\frac{\partial}{\partial \nu} = \frac{1}{2}w''' \left(\frac{\partial}{\partial z} - \nu^2 \frac{\partial}{\partial \bar{z}} + \nu \frac{\partial}{\partial t} \right).$$

The unit vector in \mathbb{R}^3 which corresponds to the point $\xi \in S^2$ is

$$e_0 = \frac{2\xi}{1 + \xi\bar{\xi}} \frac{\partial}{\partial z} + \frac{2\bar{\xi}}{1 + \xi\bar{\xi}} \frac{\partial}{\partial \bar{z}} + \frac{1 - \xi\bar{\xi}}{1 + \xi\bar{\xi}} \frac{\partial}{\partial t}.$$

The normal direction is given by the vanishing of the inner product of the preceding 2 vectors, which is easily seen to imply (for $w''' \neq 0$) $\xi = -\bar{\nu}$. At $w''' = 0$ there is an umbilic point. The remainder of the proposition follows from the incidence relation [3]:

$$\eta = \frac{1}{2}(z - 2t\xi - \bar{z}\xi^2).$$

□

The holomorphic functions of our method and that of the Weierstrass representation are related by

$$\frac{1}{(1 + \xi\bar{\xi})^2} \frac{\partial F}{\partial \xi} = \frac{1}{4} \frac{\partial^3 w}{\partial \bar{\xi}^3}.$$

4. THE GEODESIC FLOW

We now look at the metric on lagrangian sections:

Proposition 2. *The metric induced by the neutral Kähler metric on the graph of a lagrangian section $\eta = F(\xi, \bar{\xi})$ is:*

$$ds^2 = \frac{2i}{(1 + \xi\bar{\xi})^2} (\sigma_0 d\xi \otimes d\xi - \bar{\sigma}_0 d\bar{\xi} \otimes d\bar{\xi}),$$

where $\sigma_0 = -\partial\bar{F}$. Thus, for $|\sigma_0| \neq 0$ the metric is lorentz and for $|\sigma_0| = 0$ the metric is degenerate.

Proof. The neutral Kähler metric has local expression [4]:

$$\mathbb{G} = \frac{2i}{(1 + \xi\bar{\xi})^2} \left(d\eta \otimes d\bar{\xi} - d\bar{\eta} \otimes d\xi + \frac{2(\xi\bar{\eta} - \bar{\xi}\eta)}{1 + \xi\bar{\xi}} d\xi \otimes d\bar{\xi} \right). \quad (4.1)$$

We pull the metric back to the section:

$$\mathbb{G}|_{\Sigma} = \frac{2i}{(1 + \xi\bar{\xi})^2} \left[\bar{\partial}F d\bar{\xi} \otimes d\bar{\xi} - \partial\bar{F} d\xi \otimes d\xi + \left(\partial F - \bar{\partial}\bar{F} + \frac{2(\xi\bar{\eta} - \bar{\xi}\eta)}{1 + \xi\bar{\xi}} \right) d\xi \otimes d\bar{\xi} \right].$$

Now the lagrangian condition says precisely that the coefficient of the $d\xi \otimes d\bar{\xi}$ term vanishes, and the result follows. \square

We turn now to the geodesic flow. Since the metric above is flat on the normal congruence of a minimal surface, this flow is completely integrable:

Proposition 3. *Consider the normal congruence to a minimal surface $\Sigma \subset \mathbb{L}$ given by $(\xi, \eta = F(\xi, \bar{\xi}))$. The geodesic flow on Σ is completely integrable with first integrals*

$$I_1 = \frac{2i}{(1 + \xi\bar{\xi})^2} \left(\sigma_0 \dot{\xi}^2 - \bar{\sigma}_0 \dot{\bar{\xi}}^2 \right) \quad I_2 = \frac{\sigma_0^{\frac{1}{2}} \dot{\xi} + \bar{\sigma}_0^{\frac{1}{2}} \dot{\bar{\xi}}}{1 + \xi\bar{\xi}}.$$

Proof. Consider the affinely parameterised geodesic $t \mapsto (\xi(t), \eta = F(\xi(t), \bar{\xi}(t)))$ on Σ with tangent vector

$$\mathbb{T} = \dot{\xi} \frac{\partial}{\partial \xi} + \dot{\bar{\xi}} \frac{\partial}{\partial \bar{\xi}}.$$

The geodesic equation $\mathbb{T}^j \nabla_j \mathbb{T}^k = 0$, projected onto the ξ coordinate is

$$\ddot{\xi} + \Gamma_{\xi\xi}^{\xi} \dot{\xi}^2 + 2\Gamma_{\xi\bar{\xi}}^{\xi} \dot{\xi} \dot{\bar{\xi}} + \Gamma_{\bar{\xi}\bar{\xi}}^{\xi} \dot{\bar{\xi}}^2 = 0.$$

For the induced metric (as given in Proposition 2) a straight-forward calculation yields the Christoffel symbols:

$$\Gamma_{\xi\xi}^{\xi} = \frac{1}{2\sigma_0} \left(\partial\sigma_0 - \frac{2\sigma_0\bar{\xi}}{1 + \xi\bar{\xi}} \right) \quad \Gamma_{\xi\bar{\xi}}^{\bar{\xi}} = \frac{1}{2\bar{\sigma}_0} \left(\bar{\partial}\sigma_0 - \frac{2\sigma_0\xi}{1 + \xi\bar{\xi}} \right)$$

$$\Gamma_{\bar{\xi}\bar{\xi}}^{\xi} = \frac{1}{2\sigma_0} \left(\bar{\partial}\sigma_0 - \frac{2\sigma_0\xi}{1 + \xi\bar{\xi}} \right).$$

For the normal congruence of a minimal surface the holomorphic condition (2.1) implies that $\Gamma_{\xi\xi}^{\bar{\xi}} = 0$ and $\Gamma_{\xi\bar{\xi}}^{\xi} = 0$. Thus the geodesic equation reduces to

$$\ddot{\xi} = -\frac{1}{2} \partial \left[\ln \left(\frac{\sigma_0}{(1 + \xi\bar{\xi})^2} \right) \right] \dot{\xi}^2.$$

The fact that I_1 is constant along a geodesic comes from the fact that the geodesic flow preserves the length of the tangent vector \mathbb{T}^j . On the other hand,

differentiating I_2 with respect to t :

$$\begin{aligned}
\dot{I}_2 &= \frac{1}{2} \left(\frac{\sigma_0}{(1 + \xi\bar{\xi})^2} \right)^{-\frac{1}{2}} \left[\partial \left(\frac{\sigma_0}{(1 + \xi\bar{\xi})^2} \right) \dot{\xi}^2 + \bar{\partial} \left(\frac{\sigma_0}{(1 + \xi\bar{\xi})^2} \right) \dot{\xi}\dot{\bar{\xi}} \right] \\
&\quad + \frac{1}{2} \left(\frac{\bar{\sigma}_0}{(1 + \xi\bar{\xi})^2} \right)^{-\frac{1}{2}} \left[\partial \left(\frac{\bar{\sigma}_0}{(1 + \xi\bar{\xi})^2} \right) \dot{\bar{\xi}}\dot{\xi} + \bar{\partial} \left(\frac{\bar{\sigma}_0}{(1 + \xi\bar{\xi})^2} \right) \dot{\bar{\xi}}^2 \right] \\
&\quad + \frac{\sigma_0^{\frac{1}{2}}}{1 + \xi\bar{\xi}} \ddot{\xi} + \frac{\bar{\sigma}_0^{\frac{1}{2}}}{1 + \xi\bar{\xi}} \ddot{\bar{\xi}} \\
&= \frac{1}{2} \left(\frac{\sigma_0}{(1 + \xi\bar{\xi})^2} \right)^{-\frac{1}{2}} \partial \left(\frac{\sigma_0}{(1 + \xi\bar{\xi})^2} \right) \dot{\xi}^2 + \frac{1}{2} \left(\frac{\bar{\sigma}_0}{(1 + \xi\bar{\xi})^2} \right)^{-\frac{1}{2}} \bar{\partial} \left(\frac{\bar{\sigma}_0}{(1 + \xi\bar{\xi})^2} \right) \dot{\bar{\xi}}^2 \\
&\quad - \frac{1}{2} \frac{\sigma_0^{\frac{1}{2}}}{1 + \xi\bar{\xi}} \partial \left[\ln \left(\frac{\sigma_0}{(1 + \xi\bar{\xi})^2} \right) \right] \dot{\xi}^2 - \frac{1}{2} \frac{\bar{\sigma}_0^{\frac{1}{2}}}{1 + \xi\bar{\xi}} \bar{\partial} \left[\ln \left(\frac{\bar{\sigma}_0}{(1 + \xi\bar{\xi})^2} \right) \right] \dot{\bar{\xi}}^2 \\
&= 0,
\end{aligned}$$

as claimed. \square

5. EXAMPLES: THE PURE HARMONICS

We now consider the geodesic flow for the pure harmonics, that is, the minimal surfaces with

$$\frac{\partial \bar{F}}{(1 + \xi\bar{\xi})^2} = \alpha_N \xi^N,$$

for some $N \in \mathbb{N}$. These have isolated umbilic points of index $-N < 0$ at $\xi = 0$, which is also an $N + 1$ –fold branch point. By a rotation we can make α_N real and rescaling the first integrals we will set it to 1.

By Proposition 3 above, the first integrals are

$$I_1 = 2i \left(\xi^N \dot{\xi}^2 - \bar{\xi}^N \dot{\bar{\xi}}^2 \right) \quad I_2 = \xi^{N/2} \dot{\xi} + \bar{\xi}^{N/2} \dot{\bar{\xi}}.$$

These can be integrated to:

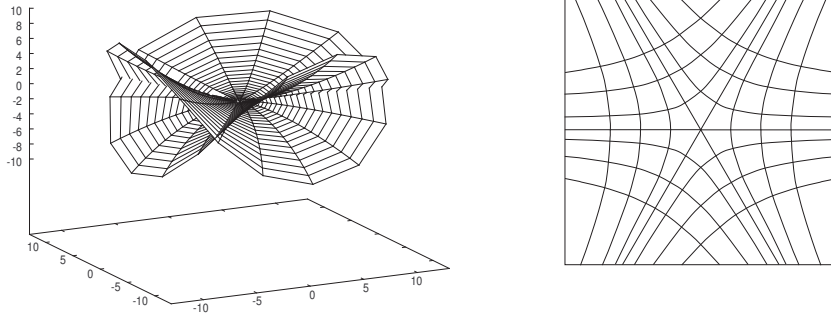
$$\frac{4iI_2}{N+2} \left(\xi^{\frac{N+2}{2}} - \bar{\xi}^{\frac{N+2}{2}} \right) = I_1 t + c_1 \quad \frac{2}{N+2} \left(\xi^{\frac{N+2}{2}} + \bar{\xi}^{\frac{N+2}{2}} \right) = I_2 t + c_2,$$

for real constants of integration c_1 and c_2 .

For null geodesics $I_1 = 0$, and if we let $\xi = Re^{i\theta}$ we get two sets of null geodesics (future- and past-directed) which are given implicitly by:

$$R^{\frac{N+2}{2}} \sin \left(\frac{N+2}{2} \theta \right) = c_1 \quad R^{\frac{N+2}{2}} \cos \left(\frac{N+2}{2} \theta \right) = c_2.$$

For $N = 0$, these form a rectangular grid, while for $N > 0$ they form the standard index $-N$ foliation about the origin. The diagram below shows the $N = 1$ minimal surface, and the foliation of null geodesics about the index -1 umbilic.



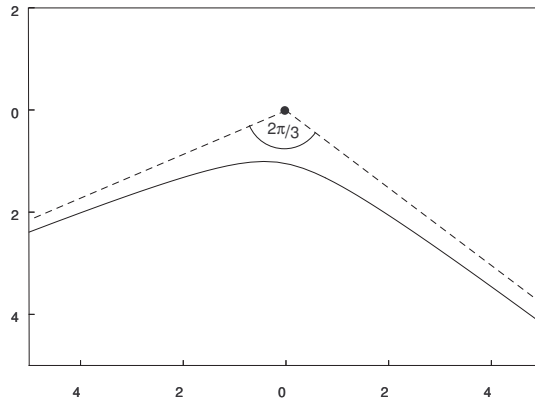
For non-null geodesics $I_1 \neq 0$ and $I_2 \neq 0$. Then the geodesics can be written parametrically:

$$R^{\frac{N+2}{2}} \sin\left(\frac{N+2}{2}\theta\right) = -\frac{N+2}{8I_2} (I_1 t + c_1) \quad R^{\frac{N+2}{2}} \cos\left(\frac{N+2}{2}\theta\right) = \frac{N+2}{4} (I_2 t + c_2).$$

The umbilic acts as a source of repulsion and the scattering angle can be found by noting that

$$\tan\left(\frac{N+2}{2}\theta\right) = -\frac{1}{2I_2} \frac{I_1 t + c_1}{I_2 t + c_2}.$$

Thus, as $t \rightarrow \pm\infty$ we have $\tan\left(\frac{N+2}{2}\theta\right) \rightarrow -\frac{I_1}{2I_2}$. We deduce then that the scattering angle is $\frac{2\pi}{N+2}$. The diagram below illustrates the scattering angle for a non-null geodesic about the $N = 1$ umbilic.



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