2022-2

A New Geometric Structure on Tangent Bundles

Nikos Georgiou

Department of Mathematics, Waterford Institute of Technology, Waterford, Co. Waterford, Ireland,

ngeorgiou@wit.ie

Brendan Guilfoyle

School of STEM, Munster Technological University, Kerry, Tralee, Co. Kerry, Ireland,

brendan.guilfoyle@mtu.ie

Recommended Citation

Nikos Georgiou, Brendan Guilfoyle, A new geometric structure on tangent bundles, Journal of Geometry and Physics, Volume 172, 2022, 104415, ISSN 0393-0440, https://doi.org/10.1016/j.geomphys.2021.104415

Journal of Geometry and Physics 172 (2022) 104415

Contents lists available at ScienceDirect

Journal of Geometry and Physics

www.elsevier.com/locate/geomphys

A new geometric structure on tangent bundles

Nikos Georgiou [a,*], Brendan Guilfoyle [b]

[a] Department of Mathematics, Waterford Institute of Technology, Waterford, Co. Waterford, Ireland
[b] School of STEM, Munster Technological University, Kerry, Tralee, Co. Kerry, Ireland

ARTICLE INFO

Article history:
Received 23 May 2021
Received in revised form 3 November 2021
Accepted 5 November 2021
Available online 12 November 2021

Keywords:
Almost para Kaehler structure
Tangent bundle
Neutral metric

ABSTRACT

For a Riemannian manifold (N, g), we construct a scalar flat neutral metric G on the tangent bundle TN. The metric is locally conformally flat if and only if either N is a 2-dimensional manifold or (N, g) is a real space form. It is also shown that G is locally symmetric if and only if g is locally symmetric. We then study submanifolds in TN and, in particular, find the conditions for a curve to be geodesic. The conditions for a Lagrangian graph in the tangent bundle TN to have parallel mean curvature are studied. Finally, using the cross product in $\mathbb{R}^3$ we show that the space of oriented lines in $\mathbb{R}^3$ can be minimally isometrically embedded in $T\mathbb{R}^3$.

Contents

1. Introduction

The geometry of the tangent bundle TN of a Riemannian manifold (N, g) has been a topic of considerable interest for at least 60 years. In the celebrated article [14], Sasaki used the Levi-Civita connection of g to split the tangent bundle TTN of TN into a horizontal and a vertical part, constructing the first geometric structure of TN. Namely, one can obtain a splitting $TTN = HN \oplus VN$, where the subbundles HN and VN of TTN are both isomorphic to the tangent bundle TN - for more details see Section 2.

* Corresponding author.
E-mail addresses: ngeorgiou@wit.ie (N. Georgiou), brendan.guilfoyle@mtu.ie (B. Guilfoyle).

https://doi.org/10.1016/j.geomphys.2021.104415

For $\bar{X} \in TTN$, we write $\bar{X} \simeq (\Pi\bar{X}, K\bar{X})$, where $\Pi\bar{X} \in HN$ and $K\bar{X} \in VN$. Sasaki defined the following metric on TN [10]:

$$G_0(\bar{X}, \bar{Y}) = g(\Pi\bar{X}, \Pi\bar{Y}) + g(K\bar{X}, K\bar{Y}).$$

The Sasaki metric is "rigid" in the following sense: G_0 is scalar flat if and only if g is flat [11].

In the years since, several new geometries on the tangent bundle TN have been constructed using the splitting of TTN - see for example [12] and [15]. When the base manifold N carries additional structure one can define other geometries on the tangent bundle. In [3] Anciaux and Romon constructed a canonical pseudo-Riemannian metric on TN derived from a Kähler structure on N. One example, the canonical neutral metric on $T\mathbb{S}^2$ defined by the standard Kähler structure of the round 2-sphere $\mathbb{S}^2$, has been used to study classical differential geometry in $\mathbb{R}^3$ - see [1], [8] and [9].

Using the Riemannian metric g one can define a canonical symplectic structure Ω in TN by the musical isomorphism between the tangent bundle and the cotangent bundle. In this article we utilize an almost paracomplex structure J on TN compatible with Ω to construct a neutral metric G on TN. In particular, we study TN endowed with the following neutral metric G:

$$G(\bar{X}, \bar{Y}) = g(\Pi\bar{X}, K\bar{Y}) + g(K\bar{X}, \Pi\bar{Y}).$$

The neutral metric G is a natural extension of the Kähler metric constructed by Anciaux and Romon in [3] to the case where the base manifold may not admit a Kähler structure (see Proposition 5).

The purpose of this article is to study the geometric properties and submanifolds of the neutral metric G. We first prove the following:

Theorem 1. *The neutral metric G has the following four properties:*

(1) G is scalar flat,
(2) G is Einstein if and only if g is Ricci flat,
(3) G is locally conformally flat if and only if either $n = 2$ or g is of constant sectional curvature,
(4) G is locally symmetric if and only if g is locally symmetric.

The geodesics of G are characterized by our second main result:

Theorem 2. *A curve $\gamma(t) = (x(t), V(t))$ in TN is a geodesic with respect to the metric G if and only if the curve x is a geodesic on N and V is a Jacobi field along x.*

The existence of a minimal Lagrangian graph in TN, where (N, g) is a 2-dimensional Riemannian manifold, implies that g is flat [2]. A generalization of this result is given by the following:

Theorem 3. *If TN contains a Lagrangian graph with parallel mean curvature, then the neutral metric G is Ricci flat.*

It is well known that the space $\mathbb{L}(\mathbb{R}^3)$ of oriented lines in $\mathbb{R}^3$ can be identified with the tangent bundle $T\mathbb{S}^2$. Guilfoyle and Klingenberg in [8] and Salvai in [13] studied the geometry of $(\mathbb{L}(\mathbb{R}^3), \mathbb{G}, \mathbb{J})$ derived from the standard Kähler structure $(T\mathbb{S}^2, g, J)$. Using this identification we show the following:

Theorem 4. *There exists a minimal isometric embedding of $(\mathbb{L}(\mathbb{R}^3), \mathbb{G})$ in $(T\mathbb{R}^3, G)$.*

The paper is organized as follows. In the next Section the new geometry is introduced in the context of almost Kähler and almost para-Kähler structures. Theorem 1 is proven in Section 3, while proofs of Theorems 2 and 3 are contained in sections 4.1 and 4.2, respectively. Theorem 4 is proven in the final section.

2. Almost (para)Kähler structures on TN

This section contains a discussion of the natural geometric structures that arise on tangent bundles of manifolds.

Let N be an n-dimensional differentiable manifold and $\pi : TN \to N$ be the canonical projection from the tangent bundle TN to N. Define the *vertical bundle* VN as the subbundle $\mathrm{Ker}(d\pi)$ of TTN. If N is equipped with an affine connection D, then define the *horizontal bundle* HN of TTN as follows:

If $\bar{X}$ is a tangent vector of TN at (p_0, V_0), there exists a curve $a(t) = (p(t), V(t)) \subset TN$ such that $a(0) = (p_0, V_0)$ and $a'(0) = \bar{X}$.

Define the connection map $K : TTN \to TN$ by $K\bar{X} = \frac{DV}{dt}(0)$ where $\frac{D}{dt}$ is the covariant derivative along $p(t)$ (see [6] and [11] for further details). The horizontal bundle HN is simply $\mathrm{Ker}(K)$ and therefore we obtain the direct sum:

$$\bar{X} \in TTN = HN \oplus VN \simeq TN \oplus TN \ni (\Pi\bar{X}, K\bar{X}).$$

Proposition 1. *[11] Given a vector field X on (N, D) there exist unique vector fields X^h, X^v on TN such that $(\Pi X^h, KX^h) = (X, 0)$ and $(\Pi X^v, KX^v) = (0, X)$.*

In addition, if X, Y are vector fields on N, we have at $(p, V) \in TN$:

$$[X^v, Y^v] = 0, \quad [X^h, Y^v] = (D_X Y)^v \simeq (0, D_X Y), \quad [X^h, Y^h] \simeq ([X, Y], -R(X, Y)V),$$

where R denotes the curvature of D.

A Riemannian metric g on N identifies the cotangent bundle T^*N with TN by the following bundle isomorphism:

$$\mathfrak{g}(p, X) = g_p(X, .) \quad \text{for any } X \in T_pN.$$

Using the canonical projection $\pi^* : T^*N \to N$, define the *Liouville form* $\xi \in \Omega^1(T^*N)$ by:

$$\xi_{(p,\beta)}(\eta) = \beta(d\pi^*\eta) \quad \text{where, } \beta \in T_p^*N \text{ and } \eta \in T_{(p,\beta)}T^*N.$$

The derivative of the Liouville form defines a canonical symplectic structure, $\Omega_* := -d\xi$, on T^*N and using the isomorphism $\mathfrak{g}$ define the symplectic structure Ω on TN by $\Omega = \mathfrak{g}^*\Omega_*$. The symplectic structure Ω is given by

$$\Omega(\bar{X}, \bar{Y}) = g(K\bar{X}, \Pi\bar{Y}) - g(\Pi\bar{X}, K\bar{Y}).$$

An *almost complex structure* (respectively *almost paracomplex structure*) on TN is an endomorphism J of TTN such that $J^2\bar{X} = -\bar{X}$ (respectively $J^2\bar{X} = \bar{X}$ and J is not the identity), for every $\bar{X} \in TTN$ and is said to be *compatible* with Ω if $\Omega(J., J.) = \Omega(., .)$ (respectively $\Omega(J., J.) = -\Omega(., .)$). In addition, in the almost paracomplex case the $+1$ and -1 eigenspaces are both assumed to be two dimensional.

Proposition 2. *Let (N, g) be a Riemannian manifold and let J_0, J_1, J_2 be the following $(1,1)$-tensors on TN:*

$$J_0\bar{X} \simeq (K\bar{X}, \Pi\bar{X}), \qquad J_1\bar{X} \simeq (\Pi\bar{X}, -K\bar{X}), \qquad J_2\bar{X} \simeq (-K\bar{X}, \Pi\bar{X}).$$

Then J_0, J_1 are almost paracomplex structures on TN while J_2 is an almost complex structure, all compatible with Ω.

Proof. A straightforward computation shows that

$$J_0^2 = J_1^2 = \text{Id}, \qquad J_2^2 = -\text{Id}, \qquad J_0 J_1 = J_2,$$

and for any $k \neq l \in \{0, 1, 2\}$

$$J_k J_l = -J_l J_k.$$

Furthermore

$$\Omega(J_0\cdot, J_0\cdot) = \Omega(J_1\cdot, J_1\cdot) = -\Omega(J_2\cdot, J_2\cdot) = -\Omega(\cdot, \cdot),$$

which establishes the compatibility conditions. □

The 3-tuple (J_0, J_1, J_2) defines an almost para-quaternionic structure on TN. Consider now the metrics G_0, G_1 and G_2, defined by

$$G_k(\cdot, \cdot) := \Omega(\cdot, J_k\cdot), \qquad k = 0, 1, 2.$$

The metric G_2 is the Riemannian Sasaki metric, up to sign, while G_0 is the neutral Sasaki metric. The neutral metric G_1 is given by

$$G_1(\bar{X}, \bar{Y}) = g(\Pi\bar{X}, K\bar{Y}) + g(K\bar{X}, \Pi\bar{Y}). \tag{1}$$

The Sasaki metrics G_0 and G_2 are well known and have been studied extensively by numerous authors - see for example [4,10,15]. In this article we fill the gap by studying the geometry of (TN, G_1). From now on and throughout this article, we simply write G for the neutral metric G_1 and J for the almost paracomplex structure J_1.

3. Curvature of the neutral metric G

In this Section we study the main geometric properties of (TN, G) and prove Theorem 1.

Denote the Levi-Civita connection of G by ∇. For a vector field X on N we use Proposition 1 to consider the unique vector fields X^h and X^v on TN such that $\Pi X^h = X$, $KX^h = 0$ and $\Pi X^v = 0$, $KX^v = X$. We do the same for the vector fields Y, Z on N. Since all quantities of type $G(Y^h, Z^v)$ are constant on the fibres, one has that $X^v G(Y^h, Z^v) = 0$.

Using the Koszul formula and the identities of Proposition 1:

$$\begin{aligned}2G(\nabla_{\bar{X}}\bar{Y}, \bar{Z}) = &\bar{X}G(\bar{Y}, \bar{Z}) + \bar{Y}G(\bar{X}, \bar{Z}) - \bar{Z}G(\bar{X}, \bar{Y}) + G([\bar{X}, \bar{Y}], \bar{Z})\\ &-G([\bar{X}, \bar{Z}], \bar{Y}) - G([\bar{Y}, \bar{Z}], \bar{X}),\end{aligned}$$

one obtains the following

$$\nabla_{X^v} Y^v = \nabla_{X^v} Y^h = 0, \qquad \nabla_{X^h} Y^v \simeq (0, D_X Y). \tag{2}$$

Thus at $(p, V) \in TN$,

$$\begin{aligned}2G(\nabla_{X^h} Y^h, Z^h) &= X^h G(Y^h, Z^h) + Y^h G(X^h, Z^h) - Z^h G(X^h, Y^h)\\ &\quad +G([X^h, Y^h], Z^h) - G([X^h, Z^h], Y^h) - G([Y^h, Z^h], X^h)\\ &= -g(R(X, Y)V, Z) + g(R(X, Z)V, Y) + g(R(Y, Z)V, X)\end{aligned}$$

and using the first Bianchi identity we finally get

$$G(\nabla_{X^h} Y^h, Z^h) = g(R(V, X)Y, Z). \tag{3}$$

Similar calculations give

$$G(\nabla_{X^h} Y^h, Z^v) = g(D_X Y, Z). \tag{4}$$

Using (3) and (4) we obtain

$$\nabla_{X^h} Y^h(V) \simeq (D_X Y, R(V, X)Y), \tag{5}$$

where $X = \Pi X^h$ and $Y = \Pi Y^h$. Putting all of these together one finds that

$$\Pi\nabla_{\bar{X}}\bar{Y}(V) = D_{\Pi\bar{X}}\Pi\bar{Y}, \qquad K\nabla_{\bar{X}}\bar{Y}(V) = D_{\Pi\bar{X}}K\bar{Y} + R(V, \Pi\bar{X})\Pi\bar{Y}. \tag{6}$$

Proposition 3. *The Riemann curvature tensor $\overline{Rm}$ of the metric G is*

$$\begin{aligned}\overline{Rm}(\bar{X}, \bar{Y}, \bar{Z}, \bar{W})|_V = &Rm(K\bar{X}, \Pi\bar{Y}, \Pi\bar{Z}, \Pi\bar{W}) + Rm(\Pi\bar{X}, K\bar{Y}, \Pi\bar{Z}, \Pi\bar{W})\\ &+Rm(\Pi\bar{X}, \Pi\bar{Y}, K\bar{Z}, \Pi\bar{W}) + Rm(\Pi\bar{X}, \Pi\bar{Y}, \Pi\bar{Z}, K\bar{W})\\ &+g((D_V R)(\Pi\bar{X}, \Pi\bar{Y})\Pi\bar{Z}, \Pi\bar{W}),\end{aligned}$$

where Rm is the Riemann curvature tensor of g and

$$(D_u R)(v, w)(z) = D_u R(v, w)z - R(D_u v, w)z - R(v, D_u w)z - R(v, w)D_u z$$

Proof. Using the second Bianchi identity, at the point $(p, V) \in TN$:

$$\begin{aligned}\bar{R}(\bar{X}, \bar{Y})\bar{Z} &= \nabla_{\bar{X}}\nabla_{\bar{Y}}\bar{Z} - \nabla_{\bar{Y}}\nabla_{\bar{X}}\bar{Z} - \nabla_{[\bar{X},\bar{Y}]}\bar{Z}\\ &\simeq (D_{\Pi\bar{X}}D_{\Pi\bar{Y}}\Pi\bar{Z} - D_{\Pi\bar{Y}}D_{\Pi\bar{X}}\Pi\bar{Z} - D_{[\Pi\bar{X},\Pi\bar{Y}]}\Pi\bar{Z},\\ &\quad D_{\Pi\bar{X}}D_{\Pi\bar{Y}}K\bar{Z} - D_{\Pi\bar{Y}}D_{\Pi\bar{X}}K\bar{Z} - D_{[\Pi\bar{X},\Pi\bar{Y}]}K\bar{Z} + D_{\Pi\bar{X}}R(V, \Pi\bar{Y})\Pi\bar{Z}\\ &\quad +R(V, \Pi\bar{X})D_{\Pi\bar{Y}}\Pi\bar{Z}) - D_{\Pi\bar{Y}}R(V, \Pi\bar{X})\Pi\bar{Z} - R(V, \Pi\bar{Y})D_{\Pi\bar{X}}\Pi\bar{Z}\\ &\quad -R(V, D_{\Pi\bar{X}}\Pi\bar{Y})\Pi\bar{Z} + R(V, D_{\Pi\bar{Y}}\Pi\bar{X})\Pi\bar{Z})\\ &= (R(\Pi\bar{X}, \Pi\bar{Y})\Pi\bar{Z},\ R(\Pi\bar{X}, \Pi\bar{Y})K\bar{Z} + (D_V R)(\Pi\bar{X}, \Pi\bar{Y})(\Pi\bar{Z})\\ &\quad +R(D_{\Pi\bar{X}}V, \Pi\bar{Y})\Pi\bar{Z} - R(D_{\Pi\bar{Y}}V, \Pi\bar{X})\Pi\bar{Z}),\end{aligned}$$

and this completes the proof. □

We are now in position to calculate the Ricci tensor:

Proposition 4. *The Ricci tensor* $\overline{Ric}$ *of the metric* G *is given by*

$$\overline{Ric}(\bar{X},\bar{Y})=2Ric(\Pi\bar{X},\Pi\bar{Y}),$$

where Ric *denotes the Ricci tensor of* g.

Proof. Following a similar method as the proof of the main theorem of [3], consider the local orthonormal field $(e_1,\dots,e_n)$ of (N,g). Define the frame $(\bar{e}_1,\dots\bar{e}_n,\bar{e}_{n+1},\dots\bar{e}_{2n})$ of TN to be $\bar{e}_k=e_k^h$ and $\bar{e}_{n+k}=e_k^v$.

For $i,j=1,\dots,n$ we have $G_{ij}=G_{n+i,n+j}=0$, and $G_{i,n+j}=\delta_{ij}$. By Proposition 3 $\overline{Ric}(X^v,Y^v)=\overline{Ric}(X^h,Y^v)=0$.

By the same Proposition,

$$\begin{aligned}\overline{Ric}(X^h,Y^h)&=\sum_{i=1}^n(\overline{Rm}(X^h,\bar{e}_i,Y^h,\bar{e}_{n+i})+\overline{Rm}(X^h,\bar{e}_{n+i},Y^h,\bar{e}_i))\\&=\sum_{i=1}^n(Rm(X,e_i,Y,e_{n+i})+\overline{Rm}(X^h,\bar{e}_{n+i},Y^h,\bar{e}_i))\\&=2\sum_{i=1}^n Rm(X,e_i,Y,e_i)\\&=2\,Ric(X,Y),\end{aligned}$$

and this completes the proof. □

Proof of Theorem 1. We now check that the four properties hold for the metric G.

Property (1): Consider, as before, the frame $(\bar{e}_1,\dots\bar{e}_n,\bar{e}_{n+1},\dots\bar{e}_{2n})$ of TN. The scalar curvature $\bar{S}$ of the metric G is

$$\bar{S}=\sum_{i,j=1}^{2n}G^{ij}\overline{Ric}(\bar{e}_i,\bar{e}_j)=\sum_{i,j=1}^{2n}G^{ij}Ric(\Pi\bar{e}_i,\Pi\bar{e}_j)=0.$$

Property (2): This follows directly from Proposition 4.

Property (3): Let $\overline{W}$ be the Weyl tensor of G. Using Proposition 5 of [3] we have

$$\begin{aligned}\overline{W}(\bar{X},\bar{Y},\bar{Z},\bar{W})=\overline{Rm}(\bar{X},\bar{Y},\bar{Z},\bar{W})-\frac{\mathrm{Ric}(\Pi\bar{Y},\Pi\bar{W})G(\bar{X},\bar{Z})+\mathrm{Ric}(\Pi\bar{X},\Pi\bar{Z})G(\bar{Y},\bar{W})}{n-1}\\+\frac{\mathrm{Ric}(\Pi\bar{X},\Pi\bar{W})G(\bar{Y},\bar{Z})+\mathrm{Ric}(\Pi\bar{Y},\Pi\bar{Z})G(\bar{X},\bar{W})}{n-1}.\end{aligned}\tag{7}$$

Assume that G is locally conformally flat and $n\geq 3$.

Let X,Y,Z,W be vector fields on N with corresponding unique vector fields X^h,X^v, Y^h,Y^v,Z^h,Z^v,W^h,W^v on TN. Using Proposition 3, we have

$$\overline{Rm}(X^h,Y^h,Z^h,W^v)=Rm(X,Y,Z,W),$$

and (7) becomes

$$Rm(X,Y,Z,W)=\frac{1}{n-1}(\mathrm{Ric}(X,Z)g(Y,W)-\mathrm{Ric}(Y,Z)g(X,W)).$$

Thus

$$Rm(X,Y,X,Y)=\frac{1}{n-1}(\mathrm{Ric}(X,X)|Y|^2-\mathrm{Ric}(X,Y)g(X,Y)).$$

On the other hand

$$Rm(Y,X,Y,X)=\frac{1}{n-1}(\mathrm{Ric}(Y,Y)|X|^2-\mathrm{Ric}(X,Y)g(X,Y)),$$

which implies that

$$\frac{\mathrm{Ric}(X,X)}{|X|^2}=\frac{\mathrm{Ric}(Y,Y)}{|Y|^2},$$

for any vector fields X, Y on N. Thus, there exists a smooth function λ on N such that

$$\mathrm{Ric}(X,X)=\lambda|X|^2,$$

which shows that g is Einstein. Since $n\geq 3$, the function λ must be constant. Let P be the plane spanned by $\{e_1,e_2\}$. Then the sectional curvature

$$K(P)=Rm(e_1,e_2,e_2,e_1)=\frac{\lambda}{n-1},$$

which is constant.

Assume the converse, that is, $n\geq 3$ and g is of constant sectional curvature K. Thus,

$$K=\frac{R}{n(n-1)}, \tag{8}$$

where R denotes the scalar curvature. Also g is locally symmetric and therefore from Proposition 3 we have

$$\begin{aligned}\overline{Rm}(\bar X,\bar Y,\bar Z,\bar W)&=\mathrm{Rm}(K\bar X,\Pi\bar Y,\Pi\bar Z,\Pi\bar W)+\mathrm{Rm}(\Pi\bar X,K\bar Y,\Pi\bar Z,\Pi\bar W)\\&\quad+\mathrm{Rm}(\Pi\bar X,\Pi\bar Y,K\bar Z,\Pi\bar W)+\mathrm{Rm}(\Pi\bar X,\Pi\bar Y,\Pi\bar Z,K\bar W).\end{aligned} \tag{9}$$

Hence, $\overline{Rm}(X^h,Y^h,Z^h,W^h)=0$ and using (7) we have,

$$\begin{aligned}\overline{W}(X^h,Y^h,Z^h,W^h)&=-\frac{1}{n-1}\Big(\mathrm{Ric}(Y,W)G(X^h,Z^h)+\mathrm{Ric}(X,Z)G(Y^h,W^h)\Big)\\&\quad+\frac{1}{n-1}\Big(\mathrm{Ric}(Y,Z)G(X^h,W^h)+\mathrm{Ric}(X,W)G(Y^h,Z^h)\Big)=0.\end{aligned}$$

On the other hand, $\overline{Rm}(X^v,Y^v,Z^v,W^v)=0$, and since in this case all Ricci tensor components of (7) vanish, we get

$$\overline{W}(X^v,Y^v,Z^v,W^v)=0.$$

Moreover, $\overline{Rm}(X^h,Y^h,Z^v,W^v)=\overline{Rm}(X^h,Y^v,Z^h,W^v)=0$, and thus (7) yields

$$\overline{W}(X^h,Y^h,Z^v,W^v)=W(X^h,Y^v,Z^h,W^v)=0.$$

Also,

$$\begin{aligned}\overline{W}(X^h,Y^h,Z^h,W^v)&=\overline{Rm}(X^h,Y^h,Z^h,W^v)-\frac{1}{n-1}\left(\mathrm{Ric}(X,Z)g(Y,W)\right)\\&\qquad+\frac{1}{n-1}\left(\mathrm{Ric}(Y,Z)g(X,W)\right)\\&=Rm(X,Y,Z,W)-\frac{1}{n-1}\left(\mathrm{Ric}(X,Z)g(Y,W)-\mathrm{Ric}(Y,Z)g(X,W)\right)\\&=Rm(X,Y,Z,W)-\frac{R}{n(n-1)}\left(g(X,Z)g(Y,W)-g(Y,Z)g(X,W)\right).\end{aligned}$$

Since g is of constant sectional curvature K, conclude

$$Rm(X,Y,Z,W)=K(g(X,Z)g(Y,W)-g(Y,Z)g(X,W)),$$

and therefore, using (8) we have

$$\overline{W}(X^h,Y^h,Z^h,W^v)=\left(K-\frac{R}{n(n-1)}\right)(g(X,Z)g(Y,W)-g(Y,Z)g(X,W))=0.$$

Using (9) and the symmetries

$$\overline{W}(a,b,c,d)=\overline{W}(c,d,a,b)=-\overline{W}(a,b,d,c),$$

one can prove that all coefficients of the Weyl tensor vanish and that $\overline{W}=0$. Thus, for $n\geq 3$, the metric G is locally conformally flat if and only if g is of constant sectional curvature.

For $n=2$, the Riemann curvature tensor is given by

$$Rm(X,Y,Z,W)=K(g(X,Z)g(Y,W)-g(Y,Z)g(X,W)),$$

where K is the Gauss curvature of g. Hence, following a similar argument as before, one can prove that for every Riemannian 2-manifold (N, g) the neutral metric G of TN is locally conformally flat.

Property (4): Assume first that g is locally symmetric. Then for any vector fields ξ, X, Y, Z on N we have, by definition,

$$D_\xi(R(X,Y)Z) = R(D_\xi X, Y)Z + R(X, D_\xi Y)Z + R(X,Y)D_\xi Z. \tag{10}$$

Using (10), a brief computation shows

$$\begin{aligned} R(\xi, V)(R(X,Y)Z) =& R(R(\xi,V)X, Y)Z + R(X, R(\xi,V)Y)Z + \\ &+ R(X,Y)(R(\xi,V)Z). \end{aligned} \tag{11}$$

We want to prove that G is locally symmetric, that is, $\nabla\bar{R} = 0$. Proposition 3 tells us that

$$\begin{aligned} \bar{R}(\bar{X},\bar{Y})\bar{Z}|_{(p,V)} \simeq& (R(\Pi\bar{X},\Pi\bar{Y})\Pi\bar{Z}, R(K\bar{X},\Pi\bar{Y})\Pi\bar{Z} + R(\Pi\bar{X},K\bar{Y})\Pi\bar{Z} + R(\Pi\bar{X},\Pi\bar{Y})K\bar{Z} \\ &+ (D_V R)(\Pi\bar{X},\Pi\bar{Y})\Pi\bar{Z}), \end{aligned}$$

and using the fact that g is locally symmetric we have

$$\bar{R}(\bar{X},\bar{Y})\bar{Z} \simeq (R(\Pi\bar{X},\Pi\bar{Y})\Pi\bar{Z}, R(K\bar{X},\Pi\bar{Y})\Pi\bar{Z} + R(\Pi\bar{X},K\bar{Y})\Pi\bar{Z} + R(\Pi\bar{X},\Pi\bar{Y})K\bar{Z}).$$

We thus obtain the following equations

$$\begin{aligned} \bar{R}(X^v,Y^v)Z^v = \bar{R}(X^v,Y^v)Z^h = \bar{R}(X^h,Y^v)Z^v = 0, \\ \bar{R}(X^v,Y^h)Z^h \simeq (0, R(X,Y)Z), \end{aligned}$$

and

$$\bar{R}(X^h,Y^h)Z^h \simeq (R(X,Y)Z, 0).$$

Apply the relations (2) and (5) to get

$$\begin{aligned} &\nabla_{\xi^v}(\bar{R}(X^v,Y^h)Z^h) = \nabla_{\xi^h}(\bar{R}(X^v,Y^h)Z^h) = 0, \\ &\nabla_{\xi^v}(\bar{R}(X^v,Y^h)Z^h) \simeq (0, D_\xi(R(X,Y)Z)), \\ &\nabla_{\xi^h}(\bar{R}(X^h,Y^h)Z^h) \simeq (D_\xi(R(X,Y)Z), R(V,\xi)(R(X,Y)Z)), \\ &\bar{R}(\nabla_{\xi^h}X^h,Y^h)Z^h \simeq (R(D_\xi X,Y)Z, R(R(V,\xi)X,Y)Z), \\ &\bar{R}(X^h,\nabla_{\xi^h}Y^h)Z^h \simeq (R(X,D_\xi Y)Z, R(X,R(V,\xi)Y)Z), \\ &\bar{R}(X^h,Y^h)\nabla_{\xi^h}Z^h \simeq (R(X,Y)D_\xi Z, R(X,Y)R(V,\xi)Z). \end{aligned}$$

Now use all of the above relations above, together with (11) to finally obtain

$$\begin{aligned} (\nabla_{\xi^h}\bar{R})(X^h,Y^h,Z^h) = \nabla_{\xi^h}(\bar{R}(X^h,Y^h)Z^h) - \bar{R}(\nabla_{\xi^h}X^h,Y^h)Z^h - \bar{R}(X^h,\nabla_{\xi^h}Y^h)Z^h \\ -\bar{R}(X^h,Y^h)\nabla_{\xi^h}Z^h = 0. \end{aligned}$$

Similar arguments establish that

$$(\nabla_{\xi^v}\bar{R})(X^h,Y^h,Z^h) = 0,$$

showing that $\nabla\bar{R} = 0$, which means G is locally symmetric.

Conversely, assume that G is locally symmetric. Then the following holds true:

$$\nabla_{\xi^h}(\bar{R}(X^h,Y^h)Z^v) = \bar{R}(\nabla_{\xi^h}X^h,Y^h)Z^v + \bar{R}(X^h,\nabla_{\xi^h}Y^h)Z^v + \bar{R}(X^h,Y^h)\nabla_{\xi^h}Z^v,$$

implying,

$$D_\xi(R(X,Y)Z) = R(D_\xi X, Y)Z + R(X, D_\xi Y)Z + R(X,Y)D_\xi Z,$$

which means that g is locally symmetric, completing the proof of Theorem 1. □

In [3] Anciaux and Romon introduced a neutral metric on TN in the case when N is equipped with a Kähler structure (j, g). An almost complex structure $\mathbb{J}$ on TN can then be defined by

$$\mathbb{J}\bar{X} = (j\Pi\bar{X}, jK\bar{X}),$$

and $\mathbb{J}$ turns out to be integrable. One can check easily that is compatible with G, that is,

$$G(\mathbb{J}\cdot, \mathbb{J}\cdot) = G(\cdot, \cdot).$$

Moreover, $\mathbb{J}$ is parallel with respect to ∇. In fact,

$$\begin{aligned}\nabla_{\bar{X}}\mathbb{J}\bar{Y}(p, V) &= (D_{\Pi\bar{X}}j\Pi\bar{Y}, D_{\Pi\bar{X}}jK\bar{Y} + R(V, \Pi\bar{X})j\Pi\bar{Y})\\ &= (jD_{\Pi\bar{X}}\Pi\bar{Y}, j(D_{\Pi\bar{X}}K\bar{Y} + R(V, \Pi\bar{X})\Pi\bar{Y}))\\ &= \mathbb{J}(D_{\Pi\bar{X}}\Pi\bar{Y}, D_{\Pi\bar{X}}K\bar{Y} + R(V, \Pi\bar{X})\Pi\bar{Y})\\ &= \mathbb{J}\nabla_{\bar{X}}\bar{Y}(p, V).\end{aligned}$$

The complex structure $\mathbb{J}$ is compatible with Ω and together with the metric $\mathbb{G}$ given by

$$\mathbb{G}(\cdot, \cdot) = \Omega(\mathbb{J}\cdot, \cdot),$$

defines a Kähler structure $(\mathbb{G}, \Omega, \mathbb{J})$ on TN. In particular,

$$\mathbb{G}(\bar{X}, \bar{Y}) = g(K\bar{X}, j\Pi\bar{Y}) - g(\Pi\bar{X}, jK\bar{Y}). \tag{12}$$

The following Proposition shows that the neutral metric G is a generalization of $\mathbb{G}$ to manifolds not necessarily endowed with a Kähler structure.

Proposition 5. *The metrics G and $\mathbb{G}$, defined respectively in (1) and (12), are isometric.*

Proof. Let N be a smooth manifold equipped with a Kähler structure (j, g). Let G and $\mathbb{G}$ be the Kähler metrics defined as above and define the following diffeomorphism: $f : TN \to TN : (p, V) \mapsto (p, -jV)$.

If $\bar{X} \in T_{(p,V)}TN$ then $\Pi df(\bar{X}) = \Pi\bar{X}$ and using the fact that j is parallel, we have that $Kdf(\bar{X}) = -jK\bar{X}$. Thus,

$$f^*G(\bar{X}, \bar{Y}) = g(\Pi\bar{X}, -jK\bar{Y}) + g(\Pi\bar{Y}, -jK\bar{X}) = \mathbb{G}(\bar{X}, \bar{Y}),$$

which proves that f is an isometry. □

4. Submanifold theory

This Section investigates submanifold theory of (TN, G) and, in particular, geodesics and Lagrangian graphs. It contains the proofs of Theorems 2 and 3.

4.1. Geodesics

We are now in position to characterize the geodesics of the neutral metric G.

Proof of Theorem 2. Let $\bar{X}(t) := \gamma'(t)$. Then, using (6), we have $\Pi\nabla_{\gamma'}\gamma' = D_{\Pi\bar{X}}\Pi\bar{X}$, and $K\nabla_{\gamma'}\gamma' = D_{\Pi\bar{X}}K\bar{X} + R(V, \Pi\bar{X})\Pi\bar{X}$. If $\gamma(t) = (x(t), V(t))$ is a geodesic then, $D_{\Pi\bar{X}}\Pi\bar{X} = 0$, and thus $D_{x'}x' = 0$, which implies that $x(t)$ is a geodesic.

On the other hand, $K\bar{X} = D_{x'}V$ and therefore, $D_{\Pi\bar{X}}K\bar{X} = D^2_{x'}V$. Using the fact that $D_{\Pi\bar{X}}K\bar{X} + R(V, \Pi\bar{X})\Pi\bar{X} = 0$, we have $D^2_{x'}V + R(V, x')x' = 0$, which shows that V is a Jacobi field along the geodesic $x(t)$.

Conversely, when $V(t)$ is a Jacobi field along the geodesic $x(t)$ then, by the same arguments, $\gamma(t) = (x(t), V(t))$ is a geodesic. □

4.2. Graph submanifolds

Let (N, g) be an n-dimensional Riemannian manifold and U an open subset of N. A vector field V on N defines an n-dimensional submanifold $\mathbb{V} \subset TN$ which is a section of the canonical bundle $\pi : TN \to N$. Such submanifolds are immersed as graphs, that is, $\mathbb{V} = f(N)$, where $f(p) = (p, V(p))$.

Let N be a $2n$-dimensional manifold equipped with a symplectic structure Ω. An immersion $f : \Sigma^n \to N$ is said to be *Lagrangian* if $f^*\Omega = 0$.

The following statement is well known: The graph of a vector field V on an open set $U \subset N$ is Lagrangian if and only if V is locally the gradient of a real smooth function on U.

We now prove our third result.

Proof of Theorem 3. Let g be the Riemannian metric on N and $\mathbb{V}$ be the submanifold of TN obtained by the image of the graph:

$$f : U \subset N \to TN : p \mapsto (p, V(p)),$$

where V is a vector field defined on the open subset of N. The fact that f is Lagrangian implies that the almost paracomplex structure J is a bundle isomorphism between the tangent bundle $T\mathbb{V}$ and the normal bundle $N\mathbb{V}$. We then consider the Maslov 1-form η on $\mathbb{V}$ defined by,

$$\eta(\cdot) = G(J\mathbb{H}, \cdot),$$

where $\mathbb{H}$ is the mean curvature vector of f. The Lagrangian condition implies the following relation:

$$d\eta(\cdot,\cdot) = \frac{1}{2}\overline{Ric}(J\cdot,\cdot)|_{\mathbb{V}},$$

where $\overline{Ric}$ denotes the Ricci tensor of G (see [5] and [7]). Assuming that $\mathbb{H}$ is parallel, the Maslov form is closed and therefore,

$$\overline{Ric}(J\bar{X}, \bar{Y}) = 0,$$

for every tangential vector fields $\bar{X}, \bar{Y}$. If X, Y are vector fields on U, the fact that f is a graph, implies that $\Pi df(X) = X$ and $\Pi df(Y) = Y$.

On the other hand, using the definition J, we have

$$J(df(X)) = (\Pi df(X), -K df(X)).$$

Thus, Proposition 4 gives

$$\begin{aligned} 0 &= \overline{Ric}(Jdf(X), df(Y)) \\ &= \overline{Ric}((\Pi df(X), -K df(X)), (\Pi df(Y), K df(Y)) \\ &= 2Ric(\Pi df(X), \Pi df(Y) \\ &= 2Ric(X, Y), \end{aligned}$$

and Theorem 3 follows. □

Corollary 1. *Let (N, g) be a non-flat Riemannian 2-manifold and Σ a Lagrangian surface of (TN, G, Ω) which projects to a curve $\gamma \subset N$.*

Then Σ has parallel mean curvature if and only if γ is a geodesic of (N, g) and Σ consists of the lines in the fibre that are orthogonal to γ'.

Proof. Suppose that g is non-flat. Using Theorem 3, the Lagrangian surface Σ can not be the graph of a smooth function on N and therefore projects to a curve $\gamma \subset N$ (ignoring the totally degenerate case where Σ is the whole fibre). In the fibre over each point on the curve, Σ consists of a 1-dimensional submanifold.

Following a similar argument as the proof of Proposition 2.1 of [2], Σ can be parametrized by:

$$f : U \subset \mathbb{R}^2 \to TN : (s,t) \mapsto (\gamma(s), a(s)\gamma'(s) + tj\gamma'(s)),$$

where j denotes the canonical complex structure on N defined as a rotation on TN about $\pi/2$ and $\gamma = \gamma(s)$ is a curve in N. The mean curvature $\mathbb{H}$ of f is

$$\mathbb{H} = (0, k(s) j\gamma'(s)),$$

where k denotes the curvature of γ. Obviously, we have that $\nabla_{\partial_t}\mathbb{H} = 0$ and

$$\nabla_{\partial_s}\mathbb{H} = (0, -k^2\gamma' + k_s j\gamma'),$$

which shows that Σ has parallel mean curvature if and only if γ is a geodesic. □

5. Special isometric embeddings

The tangent bundle of the round sphere $\mathbb{S}^2$ is identified with the space $\mathbb{L}(\mathbb{R}^3)$ of oriented lines in the Euclidean 3-space $\mathbb{R}^3$ (for more details see [8]). Define the following metric on $T\mathbb{S}^2$:

$$\mathbb{G}_p(X,Y) = g(KX, p\times \Pi Y) - g(\Pi X, p\times KY).$$

One can remark that this metric on $\mathbb{L}(\mathbb{R}^3)$ has signature (2,2), is invariant by the canonical action of the group of rigid transformations of $\mathbb{R}^3$ and that any other metric with this property is similar to this one, in some sense (for more details, see [13]).

We now prove:

Proof of Theorem 4. Consider the round 2-sphere $\mathbb{S}^2$ and let $f: T\mathbb{S}^2 \to T\mathbb{R}^3 : (p,V)\mapsto (p, V\times p)$ be the embedding, where $\times$ is the cross product in $\mathbb{R}^3$. For $X\in T_{(p,V)}T\mathbb{S}^2$, the derivative $df(X)$ is given by

$$\Pi df(X) = \Pi X, \qquad Kdf(X) = V\times \Pi X + KX\times p. \tag{13}$$

For $X,Y\in T_{(p,V)}T\mathbb{S}^2$, we have

$$\begin{aligned}(f^*G)_{(p,V)}(X,Y) &= G_{f(p,V)}(f_*X, f_*Y)\\ &= g(\Pi df X, Kdf Y) + g(\Pi df Y, Kdf X)\\ &= g(\Pi X, -\Pi Y\times V - p\times KY) + g(\Pi Y, -\Pi X\times V - p\times KX)\\ &= -g(\Pi X, p\times KY) - g(\Pi Y, p\times KX),\end{aligned}$$

which shows that $f^*G = \mathbb{G}$ and thus f is an isometric embedding.

We now show that f is minimal. Denote by $\nabla, \overline{\nabla}$ the Levi-Civita connections of $(\mathbb{R}^3, \langle .,.\rangle)$, $(T\mathbb{R}^3, G)$, respectively and denote by $D, \overline{D}$ the Levi-Civita connections of $(\mathbb{S}^2, g)$, $(T\mathbb{S}^2, \mathbb{G})$, respectively. A brief computation gives

$$\overline{\nabla}_{dfX} dfY = (\nabla_{\Pi X}\Pi Y, -\nabla_{\Pi X}\Pi Y\times V - p\times \nabla_{\Pi X}KY - \Pi Y\times KX - \Pi X\times KY + \langle \Pi X, V\rangle\, p\times \Pi Y),$$

and using the fact that $\overline{D}_X Y = (D_{\Pi X}\Pi Y, D_{\Pi X}KY - \langle V, \Pi X\rangle\, \Pi Y)$, we have

$$\begin{aligned}df(\overline{D}_X Y) = (\nabla_{\Pi X}\Pi Y - \langle \Pi X, \Pi Y\rangle\, p, &-\nabla_{\Pi X}\Pi Y\times V + \langle \Pi X, \Pi Y\rangle\, p\times V\\ &- p\times \nabla_{\Pi X}KY + \langle V, \Pi X\rangle\, p\times \Pi Y).\end{aligned}$$

The second fundamental form h of f is given by

$$\begin{aligned}h(dfX, dfY) &= \overline{\nabla}_{dfX}dfY - df(\overline{D}_X Y)\\ &= (\langle \Pi X, \Pi Y\rangle\, p, -\langle \Pi X, \Pi Y\rangle\, p\times V - \Pi Y\times KX - \Pi X\times KY).\end{aligned}$$

Suppose that $(p,V)\in T\mathbb{S}^2$ and $|V|\neq 0$. Consider the following orthogonal basis of $T_{(p,V)}T\mathbb{S}^2$:

$$E_1 = (V, p\times V), \qquad E_2 = (p\times V, V), \qquad E_3 = (V, -p\times V), \qquad E_4 = (-p\times V, V).$$

Thus,

$$\langle E_1, E_1\rangle = -\langle E_2, E_2\rangle = -\langle E_3, E_3\rangle = \langle E_4, E_4\rangle = 2|V|^2.$$

Using (13) we have

$$\begin{aligned}df(E_1) = (V, V) \qquad df(E_2) &= (p\times V, |V|^2 p - p\times V) \qquad df(E_3) = (V, -V)\\ df(E_4) &= (-p\times V, -|V|^2 p - p\times V).\end{aligned}$$

The second fundamental form for this basis is

$$h(df(E_1), df(E_1)) = h(df(E_3), df(E_3)) = (|V|^2 p, -|V|^2 p\times V),$$

and

$$h(df(E_2), df(E_2)) = h(df(E_4), df(E_4)) = (|V|^2 p, -|V|^2 p\times V - 2V).$$

The mean curvature H of f therefore is

$$H = \frac{1}{2|V|^2}\Big(h(df(E_1), df(E_1)) - h(df(E_2), df(E_2)) - h(df(E_3), df(E_3)) + h(df(E_4), df(E_4))\Big).$$

Since $V\neq 0$ is an open and dense condition, this clearly vanishes and therefore the embedding f is minimal. □

References

[1] H. Anciaux, B. Guilfoyle, On the three-dimensional Blaschke-Lebesgue problem, Proc. Am. Math. Soc. 139 (5) (2011) 1831–1839.
[2] H. Anciaux, B. Guilfoyle, P. Romon, Minimal Lagrangian surfaces in the tangent bundle of a Riemannian surface, J. Geom. Phys. 61 (1) (2011) 237–247.
[3] H. Anciaux, P. Romon, A canonical structure on the tangent bundle of a pseudo-or para-Kähler manifold, Monatshefte Math. 174 (3) (2014) 329–355.
[4] E. Boeckx, L. Vanhecke, Geometry of Riemannian manifolds and their unit tangent sphere bundles, Publ. Math. (Debr.) 57 (3) (2000) 509–533.
[5] I. Castro, F. Torralbo, F. Urbano, On Hamiltonian stationary Lagrangian spheres in non-Einstein Kähler surfaces, Math. Z. 271 (1) (2012) 259–270.
[6] P. Dombrowski, On the geometry of the tangent bundle, J. Reine Angew. Math. 210 (1962) 73–88.
[7] N. Georgiou, On minimal Lagrangian surfaces in the product of Riemannian two manifolds, Tohoku Math. J. 67 (1) (2015) 137–152.
[8] B. Guilfoyle, W. Klingenberg, An indefinite Kähler metric on the space of oriented lines, J. Lond. Math. Soc. 72 (2) (2005) 497–509.
[9] B. Guilfoyle, W. Klingenberg, A global version of a classical result of Joachimsthal, Houst. J. Math. 45 (2) (2019) 455–467.
[10] S. Gudmundsson, E. Kappos, On the geometry of the tangent bundle with the Cheeger-Gromoll metric, Tokyo J. Math. 25 (1) (2002) 75–83.
[11] O. Kowalski, Curvature of the induced Riemannian metric on the tangent bundle of a Riemannian manifold, J. Reine Angew. Math. 250 (1971) 124–129.
[12] E. Musso, F. Tricerri, Riemannian metrics on tangent bundles, Ann. Mat. Pura Appl. 150 (1) (1988) 1–19.
[13] M. Salvai, On the geometry of the space of oriented lines of Euclidean space, Manuscr. Math. 118 (2) (2005) 181–189.
[14] S. Sasaki, On the differential geometry of tangent bundles of Riemannian manifolds, Tohoku Math. J. 10 (1958) 338–354.
[15] K. Yano, S. Ishihara, Tangent and Cotangent Bundles: Differential Geometry, Marcel Dekker Inc., New York, 1973.