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On the Geometry of Spaces of Oriented Geodesics.

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Erratum to: On the geometry of spaces of oriented geodesics

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We correct some of the entries of Table 1 in our original paper.

1. Those entries of Table 1 in the original paper listing the numbers of geometric structures on the space of geodesics $L^\epsilon(S^{p,q})$ of the 3-dimensional space $S^{p,q}$ of non-zero constant curvature with $p+q=3$ are incorrect. In this note we give the correct values and arguments. In original paper we failed to note that the $(\mathfrak{so}(V) + \mathfrak{so}(V'))$ -module $V \otimes V'$, where V, V' are 2-dimensional pseudo-Euclidian spaces, has not only the invariant (para)complex structures $J = J_V \otimes 1, J' = 1 \otimes J_{V'}$, but that also their product $J'' = JJ' = J_V \otimes J_{V'}$ is an invariant (para)complex structure. We next analyze the impact of this observation in detail and correct Table 1 accordingly. The authors would like to thank Henri Anciaux for drawing their attention to this oversight.

2. Let $E = E^{p+1,q}$ be a pseudo-Euclidean 4-dimensional vector space of signature $(p+1, q) = (4, 0), (3, 1), (2, 2)$ or $(1, 3)$ with an orthonormal basis e_1, e_2, e_3, e_4 with $e_4^2 := g(e_4, e_4) = 1$. We denote by $E' = e_4^\perp$ the orthogonal complement to e_4 . The pseudo-sphere

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Table 1 Invariant geometric structures

		Symplectic structure		Complex		Para-complex		Kähler		ParaKähler	
		Int	Non	Int	Non	Int	Non	Int	Non	Int	Non
$L^\pm(E^{p+1,q})$											
$p+1+q \neq 3$	Symmetric	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset
$p+1+q = 3$		1	\emptyset	\emptyset	\emptyset	1	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset
$L^+(S^{p,q})$											
$p+q > 3$	1	1	\emptyset	\emptyset	\emptyset	1	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset
$(p,q) = (3,0)$	\mathbb{R}	2	\emptyset	1	\emptyset	\mathbb{R}, \mathbb{R}	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset
$(p,q) = (1,2)$	\mathbb{R}	2	\emptyset	1	\emptyset	\mathbb{R}, \mathbb{R}	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset
$(p,q) = (2,1)$	\mathbb{R}	2	\emptyset	1	\emptyset	\mathbb{R}	\emptyset	\mathbb{R}	\emptyset	\emptyset	\emptyset
$L^-(S^{p,q})$											
$p+q > 3$	1	\emptyset	\emptyset	1	\emptyset	\emptyset	\emptyset	1	\emptyset	\emptyset	\emptyset
$(p,q) = (0,3)$	\mathbb{R}	2	\emptyset	1	\emptyset	\mathbb{R}	\emptyset	\mathbb{R}	\emptyset	\emptyset	\emptyset
$(p,q) = (2,1)$	\mathbb{R}	2	\emptyset	1	\emptyset	\mathbb{R}	\emptyset	\mathbb{R}	\emptyset	\emptyset	\emptyset
$(p,q) = (1,2)$	\mathbb{R}	2	\emptyset	1	\emptyset	\emptyset	\emptyset	\mathbb{R}, \mathbb{R}	\emptyset	\emptyset	\emptyset
$L(\mathbb{C}P^n)$	\mathbb{R}	2	2	\emptyset	\emptyset	2	2	\emptyset	\emptyset	\emptyset	\emptyset
$L(\mathbb{H}P^n)$	1	1	\emptyset	\emptyset	\emptyset	1	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset
$L(\mathbb{O}P^2)$	1	1	\emptyset	\emptyset	\emptyset	1	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset
$L(\mathbb{C}H^n)$	\mathbb{R}	\emptyset	\emptyset	1	1	\emptyset	\emptyset	1	1	\emptyset	\emptyset
$L(\mathbb{H}H^n)$	1	\emptyset	\emptyset	1	1	\emptyset	\emptyset	1	1	\emptyset	\emptyset

of unit vectors $S^{p,q} = \{x \in E, x^2 = 1\}$ is the orbit $S^{p,q} = SO(E)e_4$. It has the reductive decomposition

$$\mathfrak{so}(E) = \mathfrak{so}(E') + e_4 \wedge E',$$

where $\mathfrak{so}(E') \simeq \mathfrak{so}_{3,0}, \mathfrak{so}_{2,1}, \mathfrak{so}_{1,2}, \mathfrak{so}_{0,3}$, respectively. The metric of E induces an $SO(E')$ -invariant metric in the tangent space $T_0S^{p,q} = e_4 \wedge E'$. The stability subalgebra of the geodesic $\ell(e_4 \wedge e_3)$ through the origin $o = e_4 \in S^{p,q}$ in the direction $e_4 \wedge e_3 \in T_oS^{p,q} = e_4 \wedge E'$ is $\mathfrak{so}(V) \times \mathfrak{so}(V^\perp)$, where $V = \text{span}(e_1, e_2)$ and $V^\perp = \text{span}(e_3, e_4)$. The geodesic is spacelike if $\epsilon = e_3^2 = 1$ and timelike if $\epsilon = e_3^2 = -1$. The space of such geodesics is identified with $L^\epsilon(S^{p,q}) = SO(E)/SO(V) \times SO(V^\perp)$ (up to a covering).

We now denote by g_V the unique (up to a scale) metric in 2-dimensional $SO(V)$ -module V and by ω_V unique (up to scale) 2-form. Then for some positive λ , the complex structure $J_V = \lambda g_V^{-1} \circ \omega_V$ is the unique (up to sign) $\mathfrak{so}(V)$ -invariant complex structure if the metric g_V has signature $(2, 0)$ or $(0, 2)$. It is the unique (up to sign) paracomplex structure if the signature of g_V is $(1, 1)$. We use similar notations for V^\perp . The reductive decomposition of the symmetric space $L^\epsilon(S^{p,q})$ is given by

$$\mathfrak{so}(E) = (\mathfrak{so}(V) \oplus (\mathfrak{so}(V^\perp))) + V \wedge V^\perp.$$

The tangent space $T_oL^\epsilon(S^{p,q})$ is identified with $\mathfrak{m} := V \wedge V^\perp \simeq V \otimes V^\perp$ with the natural action of the isotropy algebra $\mathfrak{h} = \mathfrak{so}(V) + \mathfrak{so}(V^\perp)$. We have therefore proved the following.

Proposition *The \mathfrak{h} -invariant symmetric bilinear form (respectively, invariant 2-form) on \mathfrak{m} are of the form*

$$h_{\lambda,\mu} = \lambda\omega_V \otimes \omega_{V^\perp} + \mu g_V \otimes g_{V^\perp}, \quad \omega_{\lambda,\mu} = \lambda g_V \otimes \omega_{V^\perp} + \mu \omega_V \otimes g_{V^\perp}$$

for $\lambda, \mu \in \mathbb{R}$.

The endomorphisms $J := J_V \otimes 1, J^\perp := 1 \otimes J_{V^\perp}, J' := J \circ J^\perp = J_V \otimes J_{V^\perp}$ are invariant (para)complex structure in \mathfrak{m} . Moreover, J, J^\perp are skew-symmetric with respect to any metric $h_{\lambda,\mu}$ and the endomorphism J' is symmetric.

Since $S^{p,q}$ is an irreducible symmetric space, any invariant tensor field is parallel. We obtain

Corollary *Any non-degenerate form $h_{\lambda,\mu}$ defines an invariant pseudo-Riemannian metric, any non-degenerate 2-form $\omega_{\lambda,\mu}$ defines an invariant symplectic form and the endomorphisms J, J^\perp, J' define integrable (para)complex structures on $L^\epsilon(S^{p,q})$. Furthermore, any pair of the form $(h_{\lambda,\mu}, J)$ or $(h_{\lambda,\mu}, J^\perp)$, where $h_{\lambda,\mu}$ is as above, defines a (para)Kähler structure on \mathfrak{m} .*

3. In the following table we indicate the signature of the metrics g_V and g_{V^\perp} for different values of p and ϵ and the type of the endomorphisms $J = J_V \otimes 1, J^\perp = 1 \otimes J_{V^\perp}, J' = J_V \otimes J_{V^\perp}$ (cx = complex structure, para = paracomplex structure). Note also that there is

p	ϵ	g_V	g_{V^\perp}	J	J^\perp	J'
3	+	(2,0)	(2,0)	cx	cx	para
2	+	(1,1)	(2,0)	para	cx	cx
2	-	(2,0)	(1,1)	cx	para	cx
1	+	(0,2)	(2,0)	cx	cx	para
1	-	(1,1)	(1,1)	para	para	cx
0	-	(0,2)	(1,1)	cx	para	cx

a natural isomorphism $L^\pm(S^{p,q} = L^\mp(S_-^{q,p})$, where $S_-^{q,p} = \{x \in E^{q,p+1}, x^2 = -1\}$. We finally reproduce Table 1 in its corrected form.