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2005

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THE CASIMIR EFFECT BETWEEN NON-PARALLEL PLATES BY GEOMETRIC OPTICS

BRENDAN GUILFOYLE, WILHELM KLINGENBERG, AND SIDDHARTHA SEN

ABSTRACT. Recent work by Jaffe and Scardicchio has expressed the optical approximation to the Casimir effect as a sum over geometric quantities. The first two authors have developed a technique which uses the complex geometry of the space of oriented affine lines in \mathbb{R}^3 to describe reflection of rays off a surface. This allows the quantities in the optical approximation to the Casimir effect to be calculated.

To illustrate this we determine explicitly and in closed form the geometric optics approximation of the Casimir force between two non-parallel plates. By making one of the plates finite, we regularise the divergence that is caused by the intersection of the planes. In the parallel plate limit we prove that our expression reduces to Casimir's original result.

1. INTRODUCTION

Since its discovery over 50 years ago [2] the Casimir effect has been the subject of much research. Recently the effect has been measured in different situations [1] and, as a result, there is renewed interest in calculating the Casimir effect between different shaped boundaries [9]. A particularly elegant approximation based on geometric optics has been suggested by Jaffe and Scardicchio [7] and they have shown that between a sphere and a plane this approximation appears to work well.

Their approach considers the sum over all closed paths with n reflections between the boundaries. The Dirichlet Casimir energy in this approximation is

$$\mathcal{E} = -\frac{\hbar c}{2\pi^2} \sum_n (-1)^n \iiint_{\mathcal{D}_n} \frac{\sqrt{\Delta_n(x, x)}}{l_n^3(x)} d^3x, \quad (1.1)$$

where l_n is the length and Δ_n the Van Vleck determinant of a closed n -bounce path.

In this paper we present a complex geometry scheme [4] which, in principle, can be used to evaluate the Casimir effect for a range of boundaries in the approximation scheme of Jaffe *et al.* The basic ingredient of our method is a description of reflection in \mathbb{R}^3 via the space of oriented affine lines, which we identify with \mathbb{TP}^1 . In this representation, reflection consists of the combined actions of $\overline{\text{PSL}}(2, \mathbb{C})$ and fibre mappings on \mathbb{TP}^1 .

We compute the geometric approximation to the Casimir energy by iterating the above representation. The convergence of the integral is then studied through the analytic properties of the maps involved. Of particular interest is the case where we

Date: April 24th, 2004.

1991 Mathematics Subject Classification. Primary: 78A05; Secondary: 53Z05.

Key words and phrases. Casimir, geometric optics.

have translational symmetry. This reduces the computation to a planar problem which can be addressed using trigonometric functions.

We illustrate the method by evaluating the case of non-parallel plates. For plane boundaries, the Van Vleck determinant is the reciprocal of the length squared of the path, and our approach gives an explicit iteration of the classical method of images using properties of the above mentioned action on \mathbb{TP}^1 .

We first compute the integrand of (1.1) and reduce the infinite sum over closed paths to a finite sum. Let (R, ψ) be polar coordinates in the plane, where ψ is measured from the vertical. Then:

Main Theorem 1.

There exists a closed $2m$ -bounce path in a wedge of opening angle γ iff $\gamma < \frac{\pi}{2m}$. There exists at most two closed paths (traversed in opposite directions) and the total length of a closed $2m$ -bounce reflection path starting at the point $(R \sin \psi, R \cos \psi)$ is $2R|\sin(m\gamma)|$.

There exists a closed $2m + 1$ -bounce path starting at the point $(R \sin \psi, R \cos \psi)$ iff $\gamma < \frac{\pi}{2m}$ and either of the following hold: $\psi < \pi - (m + 1)\gamma$ or $\psi > m\gamma$.

There exists exactly none, one or two closed $2m + 1$ -bounce paths from a given point, according to the inequalities above.

The total length of a closed $2m + 1$ -bounce path is $2R|\cos[\psi + (m + 1)\gamma]|$ in the former case, and $2R|\cos(\psi - m\gamma)|$ in the latter case.

There remains a divergence in the expression for the Casimir energy caused by the vertex of the wedge. We remove this by restricting one of the plates to be finite. This introduces restrictions on the domain of integration \mathcal{D}_n which must be carefully dealt with separately in the even and odd bounce cases.

Consider a plate lying between radial distances $R = R_0$ and $R = R_1$ from the origin, forming an angle γ with the horizontal plane containing the origin. Define m_0 and m_1 by $\frac{\pi}{2m_0+2} \leq \gamma < \frac{\pi}{2m_0}$, and either $\cos(m_1\gamma) \leq \frac{R_0}{R_1} \leq \frac{\cos[(m_1-1)\gamma]}{\cos\gamma}$ for m_1 even, or $\frac{\cos(m_1\gamma)}{\cos\gamma} \leq \frac{R_0}{R_1} \leq \cos[(m_1-1)\gamma]$ for m_1 odd. Our result is:

Main Theorem 2.

The geometric optics approximation to the Casimir energy between a finite plate, of dimensions stated above, and an infinite plate is:

$$\mathcal{E} = 2 \sum_{m=1}^{m_0} \mathcal{E}_{2m}^1,$$

when $m_0 \leq m_1$ and

$$\mathcal{E} = 2 \sum_{m=1}^{m_1-1} \mathcal{E}_{2m}^1 + 2\mathcal{E}_{2m_1}^0$$

when $m_0 > m_1$, where

$$\mathcal{E}_{2m}^1 = -\frac{\hbar c W \cos^2(m\gamma) \sin \gamma}{64\pi^2 \sin^4(m\gamma)} \left(\frac{1}{R_0^2 \cos \gamma} - \frac{1}{R_1^2 \cos[(m-1)\gamma] \cos(m\gamma)} \right),$$

and

$$\mathcal{E}_{2m_1}^0 = \frac{\hbar c W \cos^2(m_1\gamma)(R_0 \cos \gamma - R_1 \cos[(m_1-1)\gamma])^2}{64\pi^2 \sin^4(m_1\gamma) \cos \gamma \cos[(m_1-1)\gamma] \sin(m_1\gamma) R_0^2 R_1^2},$$

for m_1 even, and

$$\mathcal{E}_{2m_1}^0 = \frac{\hbar c W \cos^2(m_1 \gamma) (R_0 - R_1 \cos[(m_1 - 1)\gamma])^2}{64\pi^2 \sin^4(m_1 \gamma) \sin[(m_1 - 1)\gamma] \cos[(m_1 - 1)\gamma] R_0^2 R_1^2},$$

for m_1 odd.

In particular, the Casimir force is always attractive. We also prove that:

Main Theorem 3.

In the parallel plate limit the above energy reduces to Casimir's original expression:

$$\mathcal{E} = -\frac{\hbar c \pi^2 A}{1440 L^3},$$

where A is the area of the finite plate and L is the separation between the plates.

We now sketch the steps of our calculation. In Propositions 1 and 2, we reduce the infinite sum over closed paths in the Casimir energy to a finite sum and compute the total length of a closed path as a function the opening angle and initial point. This establishes Main Theorem 1.

By virtue of the general reflection formulae derived in [4] and summarised in Proposition 3, we compute the sequence of rays reflected k -times off the pair of planes. The results (Proposition 7) are then used to determine the initial direction required for the ray to return to a given point in the wedge after k -bounces (Propositions 8 and 10). In Propositions 9 and 11 we find the sequence of intersection points in the wedge. With the aid of these, we can determine the restrictions on the regions of integration in the energy for a finite plate above an infinite plane. This allows us to integrate explicitly and find the Casimir energy as a function of the dimensions and relative positions of the plates, as stated in Main Theorem 2.

This paper is organised as follows. In the next section we review the optical approximation to the Casimir effect, as developed by Jaffe and Scardicchio, and use the classical method of images to determine the existence and lengths of closed paths in an infinite wedge.

Section 3 outlines our approach to the geometry of reflection in terms of the complex geometry on \mathbb{TP}^1 - further details can be found in [4]. It also outlines how our technique can be used to compute the Casimir energy when the boundaries are not planar. In Section 4 we use this geometry to find the sequence of intersection points of a closed n -bounce path in a wedge. Here we must treat even and odd bounce closed paths separately.

Sections 5 combines these results to determine the Dirichlet Casimir energy between a plane and a finite plate. In Section 6 we obtain Casimir's original result in the parallel plate limit (Main Theorem 3).

Acknowledgements: The first two authors would like to thank Karl Luttinger for many inspiring and helpful discussions on the mathematical background of this paper. The proof of Main Theorem 1 was significantly shortened by suggestions of Patrick Dorey and Michael Farber.

The first author would like to express his appreciation for the hospitality of Grey College, Durham University, during the development of this work. This work was made possible by the International Collaboration Programme, Enterprise Ireland.

2. THE GEOMETRIC OPTICS APPROXIMATION TO THE CASIMIR EFFECT IN A WEDGE

The Casimir effect is a force that appears between conducting boundaries due to quantum fluctuations of the vacuum energy of the electromagnetic field. Recently, Jaffe and Scardicchio [7] have given the following explicit formula for the Casimir energy in terms of geometric quantities:

$$\mathcal{E} = -\frac{\hbar c}{2\pi^2} \sum_n (-1)^n \iiint_{\mathcal{D}_n} \frac{\sqrt{\Delta_n(x, x)}}{l_n^3(x)} d^3x.$$

Here the sum is over straight paths with n -bounces between the boundaries which begin and end at the point x , l_n is the length of such a path and $\Delta_n(x, x)$ is the expansion factor, or Van Vleck determinant:

$$\Delta_n(x, x') = \lim_{\delta \rightarrow 0} \delta^2 e^{-2 \int_\delta^l H dr},$$

where H is the mean curvature of the wavefront and r is an affine parameter along the path joining x and x' .

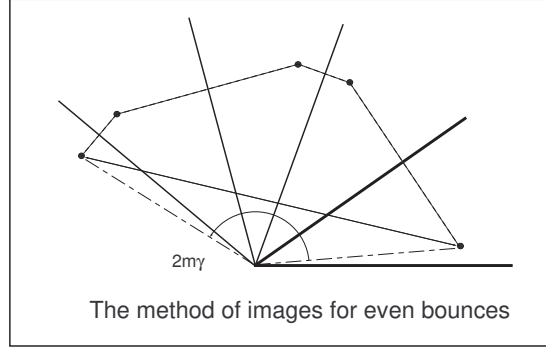
The computation of the Casimir effect can, in this approximation, be reduced to determining the length l_n , Van Vleck determinant Δ_n and region of existence \mathcal{D}_n for closed n -bounce paths between some prescribed boundary components. When the boundaries are planes the Van Vleck determinant is just the length of the path, so we need only compute l_n and \mathcal{D}_n .

We now consider the Casimir effect in a wedge formed by two non-parallel planes. Since any closed path in the wedge will be contained in a plane perpendicular to the line of intersection of the planes, the problem reduces to a 2-dimensional one.

Consider, then, a wedge in the plane with opening angle γ , where $0 < \gamma < \frac{\pi}{2}$. The following two propositions determine the existence and length of closed paths as a function of γ and the number of reflections, by applying the classical method of images.

Proposition 1. *There exists a closed $2m$ -bounce path in a wedge with angle γ iff $\gamma < \frac{\pi}{2m}$. There exists at most two closed paths (traversed in opposite directions) and the length of such a closed path is $2R|\sin(m\gamma)|$.*

Proof. We reflect the wedge $2m$ -times through one of its sides. For a point (R, ψ) in the wedge, we also reflect it $2m$ -times. The image of its path reflected in the wedge is a straight line joining the initial point to the final image point.



The angle between the ray from the origin to the beginning and endpoints is $2m\gamma$ and so the closed path exists iff $2m\gamma < \pi$, which proves the first part of the proposition. Applying the cosine rule we find that the distance between these two points, which is equal to the length of the closed path, is $2R|\sin(m\gamma)|$, as claimed. \square

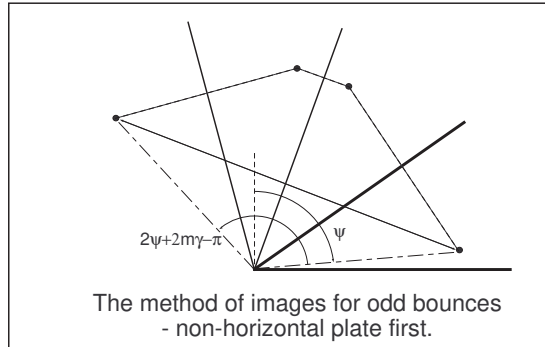
The existence and length of a closed odd-bounce path is somewhat different. Let (R, ψ) be polar coordinates in the plane, where R is the distance from the vertex, the bottom plane is aligned with the horizontal and ψ measures the angle from the vertical.

Proposition 2. *There exists a closed $2m - 1$ -bounce path starting at the point $(R \sin \psi, R \cos \psi)$ in a wedge of opening angle γ iff $\gamma < \frac{\pi}{2m-2}$ and either of the following hold: $\psi < \pi - m\gamma$ or $\psi > (m-1)\gamma$. The former applies to paths that first strike the non-horizontal plane, while the latter applies to paths that first strike the horizontal plane.*

There exists exactly none, one or two closed $2m - 1$ -bounce paths from a given point, according to the inequalities above.

The total length of a closed $2m - 1$ reflection path in a wedge with angle γ starting at the point $(R \sin \psi, R \cos \psi)$ is either $2R|\cos(\psi + m\gamma)|$ or $2R|\cos[\psi - (m-1)\gamma]|$ (respectively).

Proof. Applying the classical method of images again, we reflect the wedge $2m - 1$ -times through one of its sides. For a point (R, ψ) in the wedge, we also reflect it $2m - 1$ -times in the non-horizontal plane first. The image of its path reflected in the wedge is a straight line joining the initial point to the final image point.



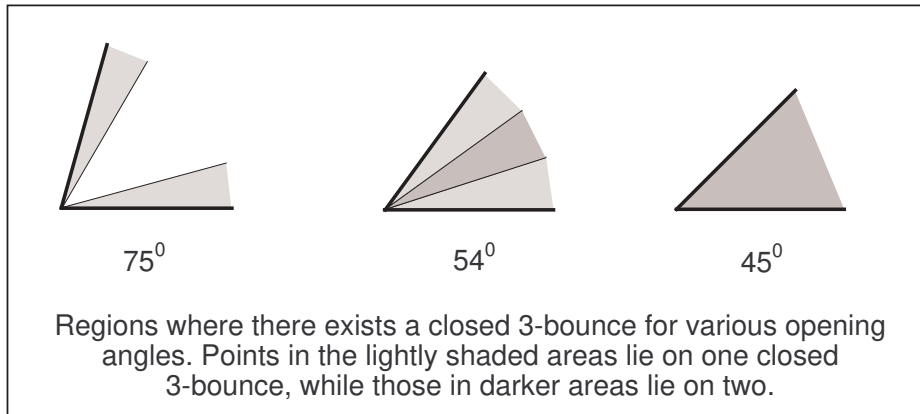
The angle between the beginning and endpoints is $2\psi - \pi + 2m\gamma$ and so the closed path exists iff $2\psi - \pi + 2m\gamma < \pi$, which proves the first part of the proposition. Applying the cosine rule we find that the distance between the two points, which is equal to the length of the closed path, is $2R|\cos(\psi + m\gamma)|$, as claimed.

Similarly for the path that first strikes the horizontal plane. □

Corollary 1. *A closed odd bounce path retraces itself.*

Proof. It is clear from symmetry that a reflected odd bounce path crosses the middle plane image at right angles, and so is reflected back along itself. □

By way of example, the diagram below shows the regions where a closed 3-bounce exists for various opening angles γ .



Introducing the preceding geometric expression for the path lengths into Jaffe and Scardicchio's optical approximation for the Casimir energy, we find that for a

wedge with opening angle γ satisfying $\frac{\pi}{2m+2} \leq \gamma < \frac{\pi}{2m+2}$:

$$\begin{aligned} \mathcal{E} = & -\frac{\hbar c}{16\pi^2} \sum_{n=1}^m \iiint_{\frac{\pi}{2}-\gamma \leq \psi \leq \frac{\pi}{2}} \frac{1}{R^4 \sin^4(n\gamma)} d^3x \\ & + \frac{\hbar c}{32\pi^2} \sum_{n=1}^m \iiint_{\psi_1 < \psi < \frac{\pi}{2}} \frac{1}{R^4 \cos^4[\psi + (n+1)\gamma]} d^3x \\ & + \frac{\hbar c}{32\pi^2} \sum_{n=1}^m \iiint_{\frac{\pi}{2}-\gamma < \psi < \psi_2} \frac{1}{R^4 \cos^4(\psi - n\gamma)} d^3x, \end{aligned}$$

where $\psi_1 = \max[n\gamma, \frac{\pi}{2} - \gamma]$ and $\psi_2 = \min[\pi - (n+1)\gamma, \frac{\pi}{2}]$.

Here, we have used the fact that even bounces count twice as they can be traversed in either direction, while odd bounces give two contributions - one from reflecting off the horizontal plane first and one from reflecting off the top plane first. We denote these three contributions by \mathcal{E}_{2m} , \mathcal{E}_{2m+1}^H and \mathcal{E}_{2m+1}^T , respectively.

The 1-bounce paths have been excluded from the sum as the energy density diverges for such paths, since their lengths go to zero as one approaches the boundary.

The difficulty with the above integrals is that they are divergent at the limit $R = 0$. This is caused by the intersection of the two planes where the closed path lengths go to zero. Indeed, we cannot expect the geometric optics approximation to be accurate near corners.

In order to remove this difficulty we put a finite separation between the two plates. This will alter the regions of integration \mathcal{D}_n and this we investigate after introducing some complex geometry in the next section.

3. THE GEOMETRY OF REFLECTION

We now describe geometric optics with the aid of the space of oriented affine lines in \mathbb{R}^3 . Further details of this approach, which we only summarise below, can be found in [4] [5].

The space of oriented affine lines in \mathbb{R}^3 can be identified with the tangent bundle to the 2-sphere $\mathbb{T}\mathbb{P}^1$ [6]. From our point of view, geometric optics is the study of 2-parameter families of oriented lines or *line congruences*. These lines, which we consider as a surface in the non-compact 4-manifold $\mathbb{T}\mathbb{P}^1$, form the rays of the optical system and the wavefronts of the optical system are orthogonal to the line congruence.

Not every line congruence has such orthogonal wavefronts - the lines may be twisting. Using the round metric on \mathbb{P}^1 , the canonical symplectic structure on $\mathbb{T}^*\mathbb{P}^1$ can be pulled back to $\mathbb{T}\mathbb{P}^1$. A line congruence is non-twisting iff this symplectic structure vanishes on the associated surface in $\mathbb{T}\mathbb{P}^1$, i.e. it is Lagrangian.

In addition, $\mathbb{T}\mathbb{P}^1$ carries a canonical complex structure given by rotation through 90° about the line. This preserves the tangent space to a line congruence at a given line iff the line is shearfree. In the case of a Lagrangian line congruence, the associated wavefronts have an umbilical point along this line.

We now consider reflection of a wavefront off a surface in \mathbb{R}^3 . First choose coordinates ξ on \mathbb{P}^1 by stereographic projection from the south pole of the unit sphere about the origin onto the plane through the equator, and coordinates on

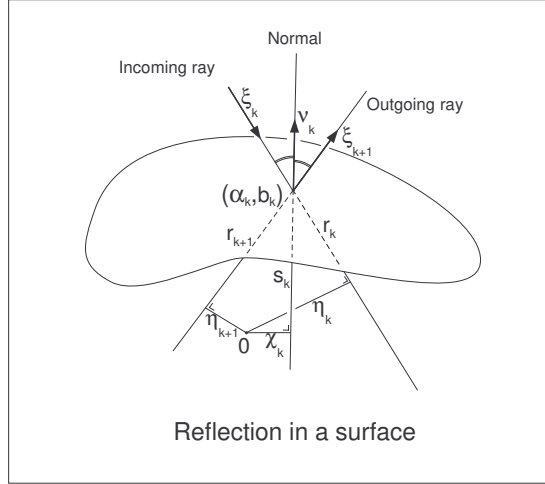
\mathbb{TP}^1 by identifying $(\xi, \eta) \in \mathbb{C}^2$ with

$$\eta \frac{\partial}{\partial \xi} + \bar{\eta} \frac{\partial}{\partial \bar{\xi}} \in T_{\xi} \mathbb{P}^1.$$

Thus ξ gives the direction of the oriented line in \mathbb{R}^3 and η gives the perpendicular distance vector from the origin to the line. Let (x^1, x^2, x^3) be Euclidean coordinates on $\mathbb{R}^3 = \mathbb{C} \oplus \mathbb{R}$, and set $z = x^1 + ix^2$, $t = x^3$. The point (z, t) lies on the line (ξ, η) iff

$$\eta = \frac{1}{2}(z - 2t\xi - \bar{z}\xi^2), \quad (3.1)$$

Consider an incoming ray $(\xi_k, \eta_k) \in \mathbb{TP}^1$ reflecting off an oriented surface at a point $(\alpha_k, b_k) \in \mathbb{C} \oplus \mathbb{R} = \mathbb{R}^3$. Suppose that the oriented normal to the surface at the point of reflection is $(\nu_k, \chi_k) \in \mathbb{TP}^1$ and that the reflected ray is $(\xi_{k+1}, \eta_{k+1}) \in \mathbb{TP}^1$. We denote the (oriented) distance of (α_k, b_k) from the closest point to the origin on the incoming ray, reflected ray and normal by r_k , r_{k+1} and s_k , respectively (see the diagram below).



The following proposition describes reflection as the combined actions of $\overline{\text{PSL}(2, \mathbb{C})}$ and fibre mappings on \mathbb{TP}^1 .

Proposition 3. [4] *The reflected ray is given by*

$$\xi_{k+1} = \frac{2\nu_k \bar{\xi}_k + 1 - \nu_k \bar{\nu}_k}{(1 - \nu_k \bar{\nu}_k) \bar{\xi}_k - 2\bar{\nu}_k}, \quad (3.2)$$

$$\eta_{k+1} = \frac{-(1 + \nu_k \bar{\nu}_k)^2 \bar{\eta}_k + (\bar{\nu}_k - \bar{\xi}_k)(1 + \nu_k \bar{\xi}_k)(1 + \nu_k \bar{\nu}_k) s_k}{((1 - \nu_k \bar{\nu}_k) \bar{\xi}_k - 2\bar{\nu}_k)^2}, \quad (3.3)$$

the distance of (α_k, b_k) from the closest point to the origin on the reflected ray is

$$r_{k+1} = r_k + \frac{2(|\nu_k - \xi_k|^2 - |1 + \nu_k \bar{\xi}_k|^2)}{(1 + \nu_k \bar{\nu}_k)(1 + \xi_k \bar{\xi}_k)} s_k, \quad (3.4)$$

and the intersection equation is

$$\eta_k = \frac{(1 + \bar{\nu}_k \xi_k)^2 \chi_k - (\nu_k - \xi_k)^2 \bar{\chi}_k + (\nu_k - \xi_k)(1 + \bar{\nu}_k \xi_k)(1 + \nu_k \bar{\nu}_k) s_k}{(1 + \nu_k \bar{\nu}_k)^2}. \quad (3.5)$$

There are many equivalent ways of rewriting and using these equations. In [4] these equations are used to explicitly determine the scattering of plane and spherical waves off planes, spheres and tori. To do this we first solve the intersection equation (3.5) to find the normal ν_k to the surface at the point of reflection and then use (3.2) and (3.3) to determine the outgoing ray.

The technique above can be utilised to compute the Casimir effect between non-planar boundaries. To get an explicit general form for the Van Vleck determinant in our formalism we proceed as follows. As detailed in [3], the mean curvature of a wavefront orthogonal to the line congruence $(\zeta, \bar{\zeta}) \rightarrow (\xi(\zeta, \bar{\zeta}), \eta(\zeta, \bar{\zeta}))$ is

$$H = \frac{\partial^+ \eta \bar{\partial} \bar{\xi} - \partial^- \eta \partial \bar{\xi}}{\partial^+ \eta \bar{\partial}^+ \eta - \partial^- \eta \bar{\partial}^- \eta},$$

where $\partial = \frac{\partial}{\partial \zeta}$ and

$$\partial^+ \eta \equiv \partial \eta - \frac{2\eta \bar{\xi} \partial \xi}{1 + \xi \bar{\xi}} + r \partial \xi, \quad \partial^- \eta \equiv \bar{\partial} \eta - \frac{2\eta \bar{\xi} \bar{\partial} \bar{\xi}}{1 + \xi \bar{\xi}} + r \bar{\partial} \bar{\xi}.$$

The Lagrangian condition for the line congruence is equivalent to H being real.

Suppose we start with a spherical wave emanating from the point $(\alpha_0, b_0) \in \mathbb{R}^3$ and this is reflected n times before passing through a point (α_{n+1}, b_{n+1}) . Let (ξ_k, η_k) be the line congruence after the k th reflection.

Proposition 4. *The geometric form of the Van Vleck determinant between the start and end points is*

$$\Delta_n(0, n+1) = \frac{1}{l_0^2} \prod_{k=1}^n \frac{[\Psi_k]_1}{[\Psi_k]_2},$$

where l_0 is the distance from the initial point to the first reflection,

$$\Psi_k = \partial^+ \eta \bar{\partial}^+ \eta - \partial^- \eta \bar{\partial}^- \eta,$$

$[\Psi_k]_1$ is evaluation of Ψ_k at the start of the k^{th} reflection and $[\Psi_k]_2$ is evaluation of Ψ_k at the end of the k^{th} reflection.

Proof. This follows immediately from the fact that the mean curvature is the derivative with respect to r of the natural log of Ψ_k . The term δ^{-2} is cancelled by the singularity in the spherical wavefront at the initial point. \square

Corollary 2. *The Van Vleck determinant of two points between plane boundaries is the reciprocal of the path length squared.*

Proof. A spherical wavefront reflected in a plane remains spherical with the same principal curvatures at the point of reflection and so $[\Psi_k]_2 = [\Psi_{k+1}]_1$. Moreover, $[\sqrt{\Psi_k}]_2 = [\sqrt{\Psi_k}]_1 + l_k$, where l_k is the length of the k^{th} reflected path. Thus

$$\Delta_n(0, n+1) = \frac{1}{[\Psi_n]_2} = \left(\sum_{k=0}^{k=n} l_k \right)^{-2},$$

as claimed. \square

For the background of this aspect of geometric optics see Chapter 5 and Appendix B of [8].

4. REFLECTIONS IN AN INFINITE WEDGE

Consider a ray originating from (α_0, b_0) and reflecting off a series of planes. Suppose that the points of reflection on the planes are given by (α_k, b_k) , the reflected directions are ξ_{k+1} and $l_{k\ k+1}$ are the (oriented) distances between these points, for $k = 1, 2, \dots$. Let ν_k be the normal direction of the k th plane.

Proposition 5. *The points of reflection are given by*

$$\alpha_k = \alpha_{k-1} + \frac{2\xi_k}{1 + \xi_k \bar{\xi}_k} l_{k-1\ k}, \quad b_k = b_{k-1} + \frac{1 - \xi_k \bar{\xi}_k}{1 + \xi_k \bar{\xi}_k} l_{k-1\ k}, \quad (4.1)$$

where

$$l_{k-1\ k} = -\frac{(1 + \xi_k \bar{\xi}_k)(\alpha_{k-1} \bar{\nu}_k + \bar{\alpha}_{k-1} \nu_k + (1 - \nu_k \bar{\nu}_k) b_{k-1} - (1 + \nu_k \bar{\nu}_k) s_k)}{|1 + \nu_k \bar{\xi}_k|^2 - |\nu_k - \xi_k|^2}. \quad (4.2)$$

Proof. The first two equations hold by the definition of $l_{k-1\ k}$ as being the distance between, and ξ_k the direction of the oriented line joining, (α_k, b_k) and (α_{k-1}, b_{k-1}) . The normal line contains the point (α_k, b_k) and the incoming ray contains the point (α_{k-1}, b_{k-1}) . The corresponding incidence relations (3.1) are

$$\chi_k = \frac{1}{2}(\alpha_k - 2b_k \nu_k - \bar{\alpha}_k \nu_k^2), \quad \eta_k = \frac{1}{2}(\alpha_{k-1} - 2b_{k-1} \xi_k - \bar{\alpha}_{k-1} \xi_k^2).$$

Introducing these relations together with the first two of the Proposition into equation (3.5) gives the third. \square

From here on we consider only the case of two plane boundaries. Consider now reflection in two planes, one of which is horizontal, both containing the origin. Then $\nu_1 = 0$ and we let ν_2 be the normal direction to the non-horizontal plane pointing inward in the first quadrant.

Proposition 6. *The lengths of the reflected paths satisfy*

$$l_{2k-1\ 2k} = -\frac{(1 + \xi_{2k} \bar{\xi}_{2k})(\alpha_{2k-1} \bar{\nu}_2 + \bar{\alpha}_{2k-1} \nu_2)}{(1 - \xi_{2k} \bar{\xi}_{2k})(1 - \nu_2 \bar{\nu}_2) + 2\xi_{2k} \bar{\nu}_2 + 2\xi_{2k} \nu_2}, \quad (4.3)$$

$$l_{2k\ 2k+1} = \frac{(1 + \xi_{2k+1} \bar{\xi}_{2k+1})(1 - \xi_{2k} \bar{\xi}_{2k})}{(1 - \xi_{2k+1} \bar{\xi}_{2k+1})(1 + \xi_{2k} \bar{\xi}_{2k})} l_{2k-1\ 2k}. \quad (4.4)$$

Proof. The first equation follows from equation (4.2) and the fact that $s_k = 0$ and $b_{2k-1} = 0$. The second equation follows from the same equation using $\nu_{2k+1} = 0$ and equation the second of (4.1) in equation (4.2). \square

In the case of two planes there is a translational symmetry which we now exploit. We assume that the line of intersection of the planes lies along the x^1 -axis and the acute angle lies in the first quadrant. Set $\nu_2 = \tan(\beta/2)$ so that $\gamma = \pi - \beta$ is the opening angle of the wedge.

With these simplifications the problem reduces to reflection in two lines in the $x^2 x^3$ -plane, and α_k, ξ_k and ν_k are all real. Let $\alpha_k = a_k \in \mathbb{R}$ and introduce polar coordinates (R, ψ) in the $x^2 x^3$ -plane.

The reflected rays after k -bounces are given by:

Proposition 7. *Consider a ray with direction ξ_1 emanating from the point (a_0, b_0) and striking the horizontal plane. The sequence of reflected rays is given by*

$$\xi_{2k} = \frac{-\sin[(k-1)\beta]\xi_1 + \cos[(k-1)\beta]}{\cos[(k-1)\beta]\xi_1 + \sin[(k-1)\beta]}, \quad \xi_{2k+1} = \frac{\cos[k\beta]\xi_1 + \sin[k\beta]}{\sin[k\beta]\xi_1 + \cos[k\beta]}, \quad (4.5)$$

$$\eta_{2k} = \frac{-(a_0 - 2b_0\xi_1 - a_0\xi_1^2)}{2(\cos[(k-1)\beta]\xi_1 + \sin[(k-1)\beta])^2}, \quad \eta_{2k+1} = \frac{a_0 - 2b_0\xi_1 - a_0\xi_1^2}{2(\sin[k\beta]\xi_1 + \cos[k\beta])^2}. \quad (4.6)$$

Proof. The first of equations (4.5) and (4.6) are true for $k = 0$. We now proceed inductively and assume they both hold for $k = k_0$. Then, since $\nu_{2k+1} = 0$ and $\nu_{2k} = \nu_2$, we have by (3.2) that

$$\xi_{2k_0+2} = \frac{1}{\xi_{2k_0+1}} = \frac{-\sin(k_0\beta)\bar{\xi}_1 + \cos(k_0\beta)}{\cos(k_0\beta)\bar{\xi}_1 + \sin(k_0\beta)},$$

which is the first of (4.5) for $k = k_0$. Similarly,

$$\begin{aligned} \xi_{2k_0+3} &= \frac{2\nu_2\bar{\xi}_{2k_0+2} + 1 - \nu_2\bar{\nu}_2}{(1 - \nu_2\bar{\nu}_2)\bar{\xi}_{2k_0+2} - 2\bar{\nu}_2} \\ &= \frac{\sin\beta[-\sin(k_0\beta)\xi_1 + \cos(k_0\beta)] + \cos\beta[\cos(k_0\beta)\xi_1 + \sin(k_0\beta)]}{\cos\beta[-\sin(k_0\beta)\xi_1 + \cos(k_0\beta)] - \sin\beta[\cos(k_0\beta)\xi_1 + \sin(k_0\beta)]} \\ &= \frac{[\cos\beta\cos(k_0\beta) - \sin\beta\sin(k_0\beta)]\xi_1 + \sin\beta\cos(k_0\beta) + \cos\beta\sin(k_0\beta)}{[-\cos\beta\sin(k_0\beta) - \sin\beta\cos(k_0\beta)]\xi_1 + \cos\beta\cos(k_0\beta) - \sin\beta\sin(k_0\beta)} \\ &= \frac{\cos[(k_0+1)\beta]\xi_1 + \sin[(k_0+1)\beta]}{-\sin[(k_0+1)\beta]\xi_1 + \cos[(k_0+1)\beta]}, \end{aligned}$$

which is the first of equation (4.5) with $k = k_0 + 1$. Thus (4.5) hold for all k .

A similar inductive argument shows that, by (3.3), equations (4.6) hold for all k . □

For future calculations we note that:

Lemma 1. *If $\xi_1 = \tan(\phi/2)$ the sequence of reflected directions satisfy*

$$\frac{1 - \xi_{2k}\bar{\xi}_{2k}}{1 + \xi_{2k}\bar{\xi}_{2k}} = -\cos[\phi + 2(k-1)\beta], \quad \frac{2\xi_{2k}}{1 + \xi_{2k}\bar{\xi}_{2k}} = \sin[\phi + 2(k-1)\beta], \quad (4.7)$$

$$\frac{1 - \xi_{2k+1}\bar{\xi}_{2k+1}}{1 + \xi_{2k+1}\bar{\xi}_{2k+1}} = \cos[\phi + 2k\beta], \quad \frac{2\xi_{2k+1}}{1 + \xi_{2k+1}\bar{\xi}_{2k+1}} = \sin[\phi + 2k\beta]. \quad (4.8)$$

Proof. These follow, with the aid of trigonometry identities, from equations (4.5). □

4.1. Even Reflections. We now study paths that return to the original point after $2m$ reflections. The direction of the initial ray is given by:

Proposition 8. *A point (a_0, b_0) lies on a closed path with $2m$ reflections iff the initial direction of the ray is*

$$\xi_1 = \frac{\sin[\psi - m\beta] \pm 1}{\cos[\psi - m\beta]}, \quad (4.9)$$

where $a_0 = R \sin \psi$ and $b_0 = R \cos \psi$. For $\xi_1 = \tan(\phi/2)$ this is equivalent to

$$\cos \phi = \mp \sin[\psi - m\beta], \quad \sin \phi = \pm \cos[\psi - m\beta]. \quad (4.10)$$

Proof. The initial point (a_0, b_0) is on a closed $2m$ -bounce if it is contained on the final outgoing ray:

$$\eta_{2m+1} = \frac{1}{2}(a_0 - 2b_0\xi_{2m+1} - a_0\xi_{2m+1}^2).$$

Substituting the first of equations (4.5) and (4.6) in this gives the quadratic equation

$$\cos[\psi - m\beta]\xi_1^2 - 2\sin[\psi - m\beta]\xi_1 - \cos[\psi - m\beta] = 0.$$

The solution to this is (4.9), or, equivalently, (4.10). \square

For future reference we note that:

Lemma 2. *For a closed $2m$ reflection path with first reflection off the horizontal plane*

$$\frac{1 - \xi_{2k}\bar{\xi}_{2k}}{1 + \xi_{2k}\bar{\xi}_{2k}} = \pm \sin[\psi - (m - 2k + 2)\beta], \quad \frac{2\xi_{2k}}{1 + \xi_{2k}\bar{\xi}_{2k}} = \pm \cos[\psi - (m - 2k + 2)\beta], \quad (4.11)$$

$$\frac{1 - \xi_{2k+1}\bar{\xi}_{2k+1}}{1 + \xi_{2k+1}\bar{\xi}_{2k+1}} = \mp \sin[\psi - (m - 2k)\beta], \quad \frac{2\xi_{2k+1}}{1 + \xi_{2k+1}\bar{\xi}_{2k+1}} = \pm \cos[\psi - (m - 2k)\beta]. \quad (4.12)$$

Proof. These follow from substituting (4.10) in (4.7) and (4.8). \square

For a closed $2m$ -bounce path, the sequence of points of intersection with the boundaries and the length of the paths are given by:

Proposition 9. *For a closed $2m$ reflection path with first reflection off the horizontal plane the sequence of points of reflection and path lengths are*

$$a_{2k} = \frac{R \cos(m\beta) \cos \beta}{\sin[\psi - (m - 2k + 1)\beta]}, \quad a_{2k-1} = \frac{R \cos(m\beta)}{\sin[\psi - (m - 2k + 2)\beta]}, \quad (4.13)$$

$$b_{2k} = -\frac{R \cos(m\beta) \sin \beta}{\sin[\psi - (m - 2k + 1)\beta]}, \quad b_{2k-1} = 0, \quad (4.14)$$

$$l_{k \ k+1} = \mp \frac{R \cos(m\beta) \sin \beta}{\sin[\psi - (m - k)\beta] \sin[\psi - (m - k + 1)\beta]}, \quad (4.15)$$

$$l_{01} = \pm \frac{R \cos \psi}{\sin[\psi - m\beta]}, \quad l_{2m \ 0} = \mp \frac{R \cos[\psi - \beta]}{\sin[\psi + (m - 1)\beta]}, \quad (4.16)$$

where the signs are chosen to make the lengths positive.

Proof. The second of (4.14) is true because odd reflections are off the horizontal plane and so have zero x^3 -coordinate.

The distance from the initial point (a_0, b_0) to the first reflection is

$$l_{01} = -\frac{R \cos \psi}{\cos \phi} = \pm \frac{R \cos \psi}{\sin[\psi - m\beta]},$$

using equation (4.10). This proves the first of (4.16). Then, by the first of (4.1),

$$a_1 = a_0 + \sin \phi l_{01} = \frac{R(\cos \phi \sin \psi - \sin \phi \cos \psi)}{\cos \phi},$$

which reduces to the second of (4.13) with $k = 1$. Next, by (4.3), (4.7) and (4.10),

$$\begin{aligned} l_{12} &= -\frac{(1 + \xi_2 \bar{\xi}_2)(\alpha_1 \bar{\nu}_2 + \bar{\alpha}_1 \nu_2)}{(1 - \xi_2 \bar{\xi}_2)(1 - \nu_2 \bar{\nu}_2) + 2\xi_2 \bar{\nu}_2 + 2\bar{\xi}_2 \nu_2} \\ &= \mp \frac{R \cos(m\beta) \sin \beta}{\sin(\psi - m\beta)[\sin(\psi - m\beta) \cos \beta + \cos(\psi - m\beta) \sin \beta]} \\ &= \mp \frac{R \cos(m\beta) \sin \beta}{\sin(\psi - m\beta) \sin[\psi - (m-1)\beta]}, \end{aligned}$$

which is (4.15) with $k = 1$. Continuing on, we have from the first of (4.1)

$$\begin{aligned} a_2 &= a_1 + \frac{2\xi_2}{1 + \xi_2 \bar{\xi}_2} l_{12} \\ &= \frac{R \cos(m\beta)}{\sin(\psi - m\beta)} - \frac{\cos(\psi - m\beta) R \cos(m\beta) \sin \beta}{\sin(\psi - m\beta) \sin[\psi - (m-1)\beta]} \\ &= \frac{R \cos(m\beta)[\sin[\psi - (m-1)\beta] - \cos(\psi - m\beta) R \cos(m\beta) \sin \beta]}{\sin(\psi - m\beta) \sin[\psi - (m-1)\beta]} \\ &= \frac{R \cos(m\beta) \cos \beta}{\sin[\psi - (m-1)\beta]}, \end{aligned}$$

which is the first of (4.13) with $k = 1$. Finally, by (4.4), together with (4.7), (4.8) and (4.10)

$$\begin{aligned} l_{23} &= -\frac{1 + \xi_3 \bar{\xi}_3}{1 - \xi_3 \bar{\xi}_3} \frac{1 - \xi_2 \bar{\xi}_2}{1 + \xi_2 \bar{\xi}_2} l_{12} \\ &= \mp \frac{\sin(\psi - m\beta)}{\sin[\psi - (m-2)\beta]} \frac{R \cos(m\beta) \sin \beta}{\sin(\psi - m\beta) \sin[\psi - (m-1)\beta]} \\ &= \mp \frac{R \cos(m\beta) \sin \beta}{\sin[\psi - (m-2)\beta] \sin[\psi - (m-1)\beta]}, \end{aligned}$$

which is (4.15) with $k = 2$.

We now proceed inductively, assuming (4.13) and (4.14) hold for $k = k_0$, and (4.15) holds for $k = 2k_0$. First, by the first of (4.1), (4.10) and (4.12)

$$\begin{aligned} a_{2k_0+1} &= a_{2k_0} + \frac{2\xi_{2k_0+1}}{1 + \xi_{2k_0+1}\bar{\xi}_{2k_0+1}} l_{2k_0 \ 2k_0+1} \\ &= \frac{R \cos(m\beta) \cos \beta}{\sin[\psi - (m - 2k_0 + 1)\beta]} - \frac{\cos[\psi - (m - 2k_0)\beta] R \cos(m\beta) \sin \beta}{\sin[\psi - (m - 2k_0)\beta] \sin[\psi - (m - 2k_0 + 1)\beta]} \\ &= \frac{R \cos(m\beta) [\cos \beta \sin[\psi - (m - 2k_0)\beta] - \cos[\psi - (m - 2k_0)\beta] \sin \beta]}{\sin[\psi - (m - 2k_0)\beta] \sin[\psi - (m - 2k_0 + 1)\beta]} \\ &= \frac{R \cos(m\beta)}{\sin[\psi - (m - 2k_0)\beta]}, \end{aligned}$$

which proves the second of (4.13) with $k = k_0 + 1$. Next we have from (4.3), (4.10) and (4.11) that

$$\begin{aligned} l_{2k_0+1 \ 2k_0+2} &= -\frac{(1 + \xi_{2k_0+2}\bar{\xi}_{2k_0+2})(a_{2k_0+1}\bar{\nu}_2 + a_{2k_0+1}\nu_2)}{(1 - \xi_{2k_0+2}\bar{\xi}_{2k_0+2})(1 - \nu_2\bar{\nu}_2) + 2\xi_{2k_0+2}\bar{\nu}_2 + 2\bar{\xi}_{2k_0+2}\nu_2} \\ &= \mp \frac{R \cos(m\beta) \sin \beta}{\sin[\psi - (m - 2k_0)\beta] [\sin[\psi - (m - 2k_0)\beta] \cos \beta + \cos[\psi - (m - 2k_0)\beta] \sin \beta]} \\ &= \mp \frac{R \cos(m\beta) \sin \beta}{\sin[\psi - (m - 2k_0)\beta] \sin[\psi - (m - 2k_0 - 1)\beta]}, \end{aligned}$$

which is (4.15) with $k = 2k_0 + 1$. Again, by the first of (4.1), (4.10) and (4.11)

$$\begin{aligned} a_{2k_0+2} &= a_{2k_0+1} + \frac{2\xi_{2k_0+2}}{1 + \xi_{2k_0+2}\bar{\xi}_{2k_0+2}} l_{2k_0+1 \ 2k_0+2} \\ &= \frac{R \cos(m\beta)}{\sin[\psi - (m - 2k_0)\beta]} - \frac{\cos[\psi - (m - 2k_0)\beta] R \cos(m\beta) \sin \beta}{\sin[\psi - (m - 2k_0)\beta] \sin[\psi - (m - 2k_0 - 1)\beta]} \\ &= \frac{R \cos(m\beta) [\sin[\psi - (m - 2k_0 - 1)\beta] - \cos[\psi - (m - 2k_0)\beta] \sin \beta]}{\sin[\psi - (m - 2k_0)\beta] \sin[\psi - (m - 2k_0 - 1)\beta]} \\ &= \frac{R \cos(m\beta) \cos \beta}{\sin[\psi - (m - 2k_0 - 1)\beta]}, \end{aligned}$$

which is the first of (4.13) with $k = k_0 + 1$. Finally, the first of (4.14) follows simply from the second of (4.1) and (4.15).

The last equation follows from (4.14) with $k = m$ and

$$l_{2m \ 0} = \mp \frac{1 + \xi_{2m+1}\bar{\xi}_{2m+1}}{1 - \xi_{2m+1}\bar{\xi}_{2m+1}} (b_0 - b_{2m}).$$

□

4.2. Odd Reflections. The direction of an initial ray that returns to the original point after $2m - 1$ reflections is given by:

Proposition 10. *A point $(a_0 = R \sin \psi, b_0 = R \cos \psi)$ is on a closed $2m - 1$ -bounce which strikes the horizontal plane first if*

$$\xi_1 = \frac{\cos[(m - 1)\beta] \pm 1}{\sin[(m - 1)\beta]}, \quad (4.17)$$

or, equivalently,

$$\sin \phi = \pm \sin[(m - 1)\beta], \quad \cos \phi = \mp \cos[(m - 1)\beta]. \quad (4.18)$$

Note that the two solutions are antipodal.

Proof. The initial point (a_0, b_0) is on a closed $2m$ -bounce if it is contained on the final outgoing ray (cf. equation (3.1)):

$$\eta_{2m} = \frac{1}{2}(a_0 - 2b_0\xi_{2m} - a_0\xi_{2m}^2).$$

Substituting the first of equations (4.5) and (4.6) in this gives the quadratic equation

$$\begin{aligned} & [\sin\psi(-1 + \cos[2(m-1)\beta]) + \cos\psi \sin[2(m-1)\beta]]\xi_1^2 \\ & + 2[\cos\psi(-1 - \cos[2(m-1)\beta]) + \sin\psi \sin[2(m-1)\beta]]\xi_1 \\ & - \sin\psi(-1 + \cos[2(m-1)\beta]) - \cos\psi \sin[2(m-1)\beta] = 0. \end{aligned}$$

The solution to this is (4.9), or, equivalently, (4.10). \square

For future use we note the following:

Lemma 3. *For a closed $2m-1$ reflection path with first reflection off the horizontal plane*

$$\frac{1 - \xi_{2k}\bar{\xi}_{2k}}{1 + \xi_{2k}\bar{\xi}_{2k}} = \pm \cos[(m-2k+1)\beta], \quad \frac{2\xi_{2k}}{1 + \xi_{2k}\bar{\xi}_{2k}} = \pm \sin[(m-2k+1)\beta], \quad (4.19)$$

$$\frac{1 - \xi_{2k+1}\bar{\xi}_{2k+1}}{1 + \xi_{2k+1}\bar{\xi}_{2k+1}} = \mp \cos[(m-2k-1)\beta], \quad \frac{2\xi_{2k+1}}{1 + \xi_{2k+1}\bar{\xi}_{2k+1}} = \pm \cos[(m-2k-1)\beta]. \quad (4.20)$$

Proof. These follow from substituting (4.18) in (4.7) and (4.8). \square

For a closed $2m-1$ -bounce path, the sequence of points of intersection with the boundaries and the length of the paths are given by:

Proposition 11. *For a closed $2m-1$ -bounce with first reflection off the horizontal plane the sequence of points of reflection and path lengths are*

$$a_{2k} = \frac{R \sin[\psi + (m-1)\beta] \cos \beta}{\cos[(m-2k)\beta]}, \quad a_{2k-1} = \frac{R \sin[\psi + (m-1)\beta]}{\cos[(m-2k+1)\beta]}, \quad (4.21)$$

$$b_{2k} = -\frac{R \sin[\psi + (m-1)\beta] \sin \beta}{\cos[(m-2k)\beta]}, \quad b_{2k-1} = 0, \quad (4.22)$$

$$l_{k+1} = \mp \frac{R \sin[\psi + (m-1)\beta] \sin \beta}{\cos[(m-k)\beta] \cos[(m-k-1)\beta]}, \quad l_{01} = \pm \frac{R \cos \psi}{\sin[\psi - m\beta]}, \quad (4.23)$$

where the signs are chosen to make the lengths positive.

Proof. The second equation of (4.22) holds since odd reflections are off the horizontal plane, so the x^3 -coordinate of the intersection point is zero.

The distance from the initial point (a_0, b_0) to the first reflection is

$$l_{01} = -\frac{R \cos \psi}{\cos \phi} = \pm \frac{R \cos \psi}{\cos[(m-1)\beta]},$$

using equation (4.18). This proves the second of (4.23). Then, by the first of (4.1) and (4.18)

$$a_1 = a_0 + \sin \phi l_{01} = \frac{R \sin[\psi + (m-1)\beta]}{\cos[(m-1)\beta]},$$

which is the second of (4.21) with $k = 1$. Next, by (4.3), (4.18) and (4.19)

$$\begin{aligned} l_{12} &= -\frac{(1 + \xi_2 \bar{\xi}_2)(a_1 \bar{\nu}_2 + a_1 \nu_2)}{(1 - \xi_2 \bar{\xi}_2)(1 - \nu_2 \bar{\nu}_2) + 2\xi_2 \bar{\nu}_2 + 2\xi_2 \nu_2} \\ &= \mp \frac{R \sin[\psi + (m-1)\beta] \sin \beta}{\cos[(m-1)\beta][\cos[(m-1)\beta] \cos \beta + \sin[(m-1)\beta] \sin \beta]} \\ &= \mp \frac{R \sin[\psi + (m-1)\beta] \sin \beta}{\cos[(m-1)\beta] \cos[(m-2)\beta]}, \end{aligned}$$

which is the first of (4.23) with $k = 1$. Continuing on, we have by the first of (4.1), (4.18) and (4.19)

$$\begin{aligned} a_2 &= a_1 + \frac{2\xi_2}{1 + \xi_2 \bar{\xi}_2} l_{12} \\ &= \frac{R \sin[\psi + (m-1)\beta][\cos[(m-2)\beta] - \sin[(m-1)\beta] \sin \beta]}{\cos[(m-1)\beta] \cos[(m-2)\beta]} \\ &= \frac{R \sin[\psi + (m-1)\beta] \cos[(m-1)\beta] \cos \beta}{\cos[(m-1)\beta] \cos[(m-2)\beta]} \\ &= \frac{R \sin[\psi + (m-1)\beta] \cos \beta}{\cos[(m-2)\beta]}, \end{aligned}$$

which is the first of (4.21) with $k = 1$. Finally, by (4.4), (4.19) and (4.20)

$$\begin{aligned} l_{23} &= -\frac{1 + \xi_3 \bar{\xi}_3}{1 - \xi_3 \bar{\xi}_3} \frac{1 - \xi_2 \bar{\xi}_2}{1 + \xi_2 \bar{\xi}_2} l_{12} \\ &= \mp \frac{\cos[(m-1)\beta]}{\cos[(m-3)\beta]} \frac{R \sin[\psi + (m-1)\beta] \sin \beta}{\cos[(m-1)\beta] \cos[(m-2)\beta]} \\ &= \mp \frac{R \sin[\psi + (m-1)\beta] \sin \beta}{\cos[(m-3)\beta] \cos[(m-2)\beta]}, \end{aligned}$$

which is the first of (4.23) with $k = 2$.

We proceed inductively, assuming (4.21) and (4.22) hold for $k = k_0$, and (4.23) holds for $k = 2k_0$. First, by the first of (4.1), (4.18) and (4.20)

$$\begin{aligned} a_{2k_0+1} &= a_{2k_0} + \frac{2\xi_{2k_0+1}}{1 + \xi_{2k_0+1} \bar{\xi}_{2k_0+1}} l_{2k_0 \ 2k_0+1} \\ &= \frac{R \sin[\psi + (m-1)\beta] \cos \beta}{\cos[(m-2k_0)\beta]} - \frac{\sin[(m-2k_0-1)\beta] R \sin[\psi + (m-1)\beta] \sin \beta}{\cos[(m-2k_0)\beta] \cos[(m-2k_0-1)\beta]} \\ &= \frac{R \sin[\psi + (m-1)\beta][\cos \beta \cos[(m-2k_0-1)\beta] - \sin[(m-2k_0-1)\beta] \sin \beta]}{\cos[(m-2k_0)\beta] \cos[(m-2k_0-1)\beta]} \\ &= \frac{R \sin[\psi + (-1)m\beta]}{\cos[(m-2k_0-1)\beta]}, \end{aligned}$$

which proves the second of (4.21) holds with $k = k_0 + 1$. Next, by (4.3), (4.18) and (4.19), we have

$$\begin{aligned} l_{2k_0+1 \ 2k_0+2} &= -\frac{(1 + \xi_{2k_0+2}\bar{\xi}_{2k_0+2})(a_{2k_0+1}\bar{\nu}_2 + a_{2k_0+1}\nu_2)}{(1 - \xi_{2k_0+2}\bar{\xi}_{2k_0+2})(1 - \nu_2\bar{\nu}_2) + 2\xi_{2k_0+2}\bar{\nu}_2 + 2\bar{\xi}_{2k_0+2}\nu_2} \\ &= \mp \frac{R \sin[\psi + (m-1)\beta] \sin \beta}{\cos[(m-2k_0-1)\beta][\cos[(2k_0-m+1)\beta] \cos \beta - \sin[(2k_0-m+1)\beta] \sin \beta]} \\ &= \mp \frac{R \sin[\psi + (m-1)\beta] \sin \beta}{\cos[(m-2k_0-1)\beta] \cos[(m-2k_0-2)\beta]}, \end{aligned}$$

which is the first of (4.23) with $k = 2k_0 + 1$. Now by the first of (4.1) and (4.19)

$$\begin{aligned} a_{2k_0+2} &= a_{2k_0+1} + \frac{2\xi_{2k_0+2}}{1 + \xi_{2k_0+2}\bar{\xi}_{2k_0+2}} l_{2k_0+1 \ 2k_0+2} \\ &= \frac{R \sin[\psi + (m-1)\beta]}{\cos[(m-2k_0-1)\beta]} - \frac{\sin[(m-2k_0-1)\beta] R \sin[\psi + (m-1)\beta] \sin \beta}{\cos[(m-2k_0-1)\beta] \cos[(m-2k_0-2)\beta]} \\ &= \frac{R \sin[\psi + (m-1)\beta][\cos[(m-2k_0-2)\beta] - \sin[(m-2k_0-1)\beta] \sin \beta]}{\cos[(m-2k_0-1)\beta] \cos[(m-2k_0-2)\beta]} \\ &= \frac{R \sin[\psi + (m-1)\beta] \cos \beta}{\cos[(m-2k_0-2)\beta]}, \end{aligned}$$

which is the first of (4.21) with $k = k_0 + 1$. Finally, (4.22) follows simply from the second of (4.1) and (4.23). \square

As proved earlier, a closed odd bounce retraces itself. This can also be seen from the fact that for a closed $2m-1$ -bounce, with m odd, $a_{2k-1} = a_{2(m-k)+1}$, while for m even $a_{2k} = a_{2(m-k)}$.

Proposition 12. *For a closed $2m-1$ -bounce with first reflection off the horizontal plane the sequence of points of reflection and path lengths are*

$$a_{2k} = \frac{R \sin[\psi - m\beta]}{\cos[(m-2k)\beta]}, \quad a_{2k-1} = \frac{R \sin[\psi - m\beta] \cos \beta}{\cos[(m-2k+1)\beta]}, \quad (4.24)$$

$$b_{2k-1} = -\frac{R \sin[\psi - m\beta] \sin \beta}{\cos[(m-2k+1)\beta]}, \quad b_{2k} = 0, \quad (4.25)$$

Proof. These follow from (4.21) and (4.22) by reflecting through the bisector of the wedge: $(R, \psi) \rightarrow (R, \beta - \psi)$, so that

$$(a_{2k}, b_{2k}) \rightarrow \left(-\frac{a_{2k}}{\cos \beta}, 0 \right), \quad (a_{2k-1}, 0) \rightarrow (-a_{2k-1} \cos \beta, a_{2k-1} \sin \beta).$$

\square

Note:

Instead of using the method of images as in Section 2, the lengths of the closed $2m$ - and $2m-1$ -bounce paths can be found from Propositions 9 and 11 and the

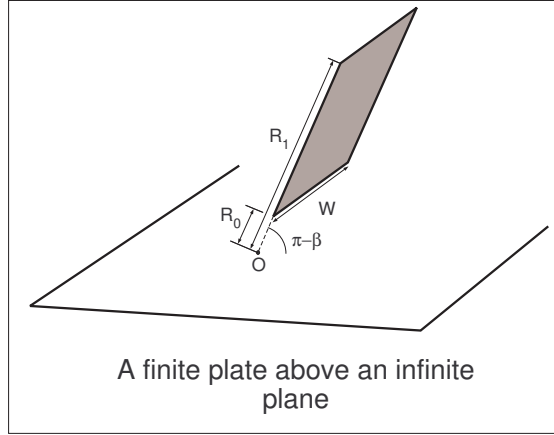
following two trigonometric identities:

$$\frac{\cos(\psi - \beta)}{\sin[\psi + (m - 1)\beta]} - \frac{\cos \psi}{\sin(\psi - m\beta)} + \sum_{k=1}^{2m-1} \frac{\cos(m\beta) \sin \beta}{\sin[\psi - (m - k)\beta] \sin[\psi - (m - k + 1)\beta]} = 2 \sin(m\beta),$$

$$\frac{\cos \psi}{\cos[(m - 1)\beta]} - \sum_{k=1}^{m-1} \frac{\sin[\psi + (m - 1)\beta] \sin \beta}{\cos[(m - k)\beta] \cos[(m - k - 1)\beta]} = 2 \cos[\psi + (m - 1)\beta].$$

5. THE CASIMIR ENERGY WITH FINITE TOP PLATE

Consider the case where the top plate is finite, lying between $R = R_0$ and $R = R_1$ where $R_1 > R_0$. Let the width of the top plate be W and the bottom plate be infinite in all directions.



The effect of limiting the upper plate is that some of the closed paths must now be excluded from the energy as they run off the edge of the plate. This restricts the domain of integration \mathcal{D}_n .

Consider a closed $2m$ - or $2m + 1$ -bounce path that first strikes the horizontal plate. The closest and furthest points from the vertex have x^2 -coordinate:

| | Closest | Furthest |
|--------|-----------|----------|
| m odd | a_{m+1} | a_{2m} |
| m even | a_m | a_{2m} |

For a $2m + 1$ -bounce path that strikes the top plate first, the closest and furthest points from the vertex have x^2 -coordinate:

| | Closest | Furthest |
|--------|-----------|----------|
| m odd | a_m | a_1 |
| m even | a_{m+1} | a_1 |

We will use these to restrict the domains of integration, taking the odd and even bounce cases separately.

5.1. The Even Bounce Contribution. We start by considering the case of a $2m$ -bounce. The restrictions we must introduce for the finite plate, obtained from (4.13) are

$$\begin{array}{c|c|c} \text{m odd} & a_{m+1} \geq -R_0 \cos \beta & a_{2m} \leq -R_1 \cos \beta \\ \text{m even} & a_m \geq -R_0 \cos \beta & a_{2m} \leq -R_1 \cos \beta \end{array}$$

Suppose that $m = 2n$, then the restrictions are

$$a_m = a_{2n} = \frac{R \cos(2n\beta) \cos \beta}{\sin(\psi - \beta)} \geq -R_0 \cos \beta, \quad (5.1)$$

$$a_{2m} = a_{4n} = \frac{R \cos(2n\beta) \cos \beta}{\sin[\psi + (2n - 1)\beta]} \leq -R_1 \cos \beta, \quad (5.2)$$

which we put together to find

$$-\frac{R_0 \sin(\psi - \beta)}{\cos(2n\beta)} \leq R \leq -\frac{R_1 \sin[\psi + (2n - 1)\beta]}{\cos(2n\beta)}.$$

We now consider the restriction this inequality places on ψ , namely:

$$-R_0 \sin(\psi - \beta) \leq -R_1 \sin[\psi + (2n - 1)\beta].$$

Since $\beta - \frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}$ and $\beta > \frac{4n-1}{4n}\pi$ the function

$$f(\psi) = -R_0 \sin(\psi - \beta) + R_1 \sin[\psi + (2n - 1)\beta],$$

is a decreasing function of ψ . Thus the inequality holds iff $f(\frac{\pi}{2}) \leq 0$, or $-R_0 \cos \beta \leq -R_1 \cos[(2n - 1)\beta]$.

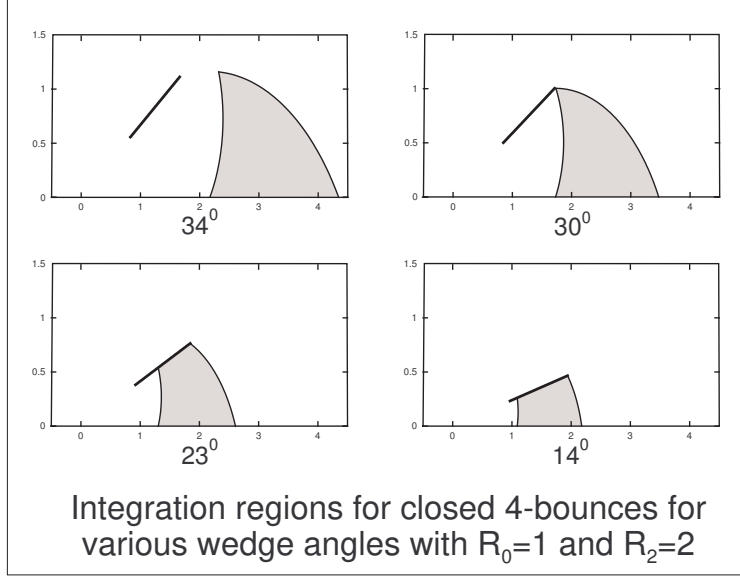
We consider two cases: either $f(\beta - \frac{\pi}{2}) \leq 0$, i.e. $R_0 \leq R_1 \cos(2n\beta)$, or there exists $\psi_0 \in [\beta - \frac{\pi}{2}, \frac{\pi}{2}]$ such that $f(\psi_0) = 0$:

$$-R_0 \sin(\psi_0 - \beta) = -R_1 \sin[\psi_0 + (2n - 1)\beta].$$

This can also be written

$$\tan \psi_0 = \frac{R_0 \sin \beta + R_1 \sin[(2n - 1)\beta]}{R_0 \cos \beta - R_1 \cos[(2n - 1)\beta]}. \quad (5.3)$$

In the former case the region of integration is $[\beta - \frac{\pi}{2}, \frac{\pi}{2}]$ while in the latter case it is $[\psi_0, \frac{\pi}{2}]$. The diagram below shows the regions of integration for a closed 4-bounce for various wedge angles when $R_0 = 1$ and $R_1 = 2$. Here we can see the transition from the latter to the former integration regions occuring at 30° (note that, for illustration purposes, the scales on the two axes are not equal).



In fact, it is easy to see that the regions of integration lie between two circles which pass through the origin and whose second point of intersection lies in the first quadrant at an angle ψ_0 . This follows from studying equations (5.1) and (5.2). The region of integration is also seen to lie within the wedge.

If we let $\psi_1 = \max[\psi_0, \beta - \frac{\pi}{2}]$ then the energy associated with a closed $4n$ -bounce can be computed:

$$\begin{aligned}
\mathcal{E}_{4n} &= -\frac{\hbar c W}{32\pi^2} \int_{\psi_1}^{\frac{\pi}{2}} \int_{-\frac{R_0 \sin(\psi - \beta)}{\cos(2n\beta)}}^{-\frac{R_1 \sin(\psi + (2n-1)\beta)}{\cos(2n\beta)}} \frac{1}{R^3 \sin^4(2n\beta)} dR d\psi \\
&= \frac{\hbar c W}{64\pi^2 \sin^4(2n\beta)} \int_{\psi_1}^{\frac{\pi}{2}} \frac{\cos^2(2n\beta)}{R_1^2 \sin^2[\psi + (2n-1)\beta]} - \frac{\cos^2(2n\beta)}{R_0^2 \sin^2(\psi - \beta)} d\psi \\
&= -\frac{\hbar c W \cos^2(2n\beta)}{64\pi^2 \sin^4(2n\beta)} \left[\frac{\cot[\psi + (2n-1)\beta]}{R_1^2} - \frac{\cot(\psi - \beta)}{R_0^2} \right]_{\psi_1}^{\frac{\pi}{2}} \\
&= -\frac{\hbar c W \cos^2(2n\beta) \cos \psi_1}{64\pi^2 \sin^4(2n\beta)} \left(\frac{1}{R_0^2 \cos \beta \sin(\psi_1 - \beta)} - \frac{1}{R_1^2 \cos[(2n-1)\beta] \sin[\psi_1 + (2n-1)\beta]} \right).
\end{aligned}$$

Now suppose that $m = 2n - 1$. The restrictions together with (4.13), lead to

$$\begin{aligned}
a_{m+1} = a_{2n} &= \frac{R \cos[(2n-1)\beta] \cos \beta}{\sin \psi} \geq -R_0 \cos \beta, \\
a_{2m} = a_{4n-2} &= \frac{R \cos[(2n-1)\beta] \cos \beta}{\sin[\psi + (2n-2)\beta]} \leq -R_1 \cos \beta,
\end{aligned}$$

which we put together to find

$$-\frac{R_0 \sin \psi}{\cos[(2n-1)\beta]} \leq R \leq -\frac{R_1 \sin[\psi + (2n-2)\beta]}{\cos[(2n-1)\beta]}.$$

Thus this places the following inequality on ψ

$$R_0 \sin \psi \leq R_1 \sin[\psi + (2n - 2)\beta],$$

or

$$\tan \psi_0 = \frac{R_1 \sin[(2n - 2)\beta]}{R_0 - R_1 \cos[(2n - 2)\beta]}. \quad (5.4)$$

A similar monotone argument to the $m = 2n$ case shows that the inequality holds iff $R_0 \leq R_1 \cos[(2n - 2)\beta]$.

We consider two cases: either

$$-R_0 \cos \beta \leq -R_1 \cos[(2n - 1)\beta],$$

or there exists $\psi_0 \in [\beta - \frac{\pi}{2}, \frac{\pi}{2}]$ such that

$$R_0 \sin \psi_0 = R_1 \sin[\psi_0 + (2n - 2)\beta].$$

In the former case the region of integration is $[\beta - \frac{\pi}{2}, \frac{\pi}{2}]$ while in the latter case it is $[\psi_0, \frac{\pi}{2}]$. As before, these lie between two circles which intersect at the origin and at a point in the first quadrant.

Letting $\psi_1 = \max[\psi_0, \beta - \frac{\pi}{2}]$ the resulting energy integrates up to

$$\mathcal{E}_{4n-2} = -\frac{\hbar c W \cos^2[(2n - 1)\beta] \cos \psi_1}{64\pi^2 \sin^4[(2n - 1)\beta]} \left(\frac{1}{R_0^2 \sin \psi_1} - \frac{1}{R_1^2 \cos[(2n - 2)\beta] \sin[\psi_1 + (2n - 2)\beta]} \right).$$

We combine these results for the even bounce case:

Proposition 13. *Given β , R_0 and R_1 , define m_0 and m_1 by $\frac{2m_0-1}{2m_0}\pi < \beta \leq \frac{2m_0+1}{2m_0+2}\pi$, and either $\cos(m_1\beta) \leq \frac{R_0}{R_1} \leq \frac{\cos[(m_1-1)\beta]}{\cos \beta}$ for m_1 even or $\frac{\cos(m_1\beta)}{\cos \beta} \leq \frac{R_0}{R_1} \leq \cos[(m_1-1)\beta]$ for m_1 odd.*

Then the even contribution to the Casimir energy is

$$\mathcal{E}_{\text{even}} = 2 \sum_{m=1}^{m_0} \mathcal{E}_{2m}^1,$$

when $m_0 \leq m_1$ and

$$\mathcal{E}_{\text{even}} = 2 \sum_{m=1}^{m_1-1} \mathcal{E}_{2m}^1 + 2\mathcal{E}_{2m_1}^0,$$

when $m_0 > m_1$, where

$$\mathcal{E}_{2m}^1 = \frac{\hbar c W \cos^2(m\beta) \sin \beta}{64\pi^2 \sin^4(m\beta)} \left(\frac{1}{R_0^2 \cos \beta} - \frac{1}{R_1^2 \cos[(m-1)\beta] \cos(m\beta)} \right), \quad (5.5)$$

and

$$\mathcal{E}_{2m}^0 = -\frac{\hbar c W \cos^2(m\beta) (R_0 \cos \beta - R_1 \cos[(m-1)\beta])^2}{64\pi^2 \sin^4(m\beta) \cos \beta \cos[(m-1)\beta] \sin(m\beta) R_0^2 R_1^2}, \quad (5.6)$$

for m_1 even, and

$$\mathcal{E}_{2m}^0 = -\frac{\hbar c W \cos^2(m\beta) (R_0 - R_1 \cos[(m-1)\beta])^2}{64\pi^2 \sin^4(m\beta) \sin[(m-1)\beta] \cos[(m-1)\beta] R_0^2 R_1^2}, \quad (5.7)$$

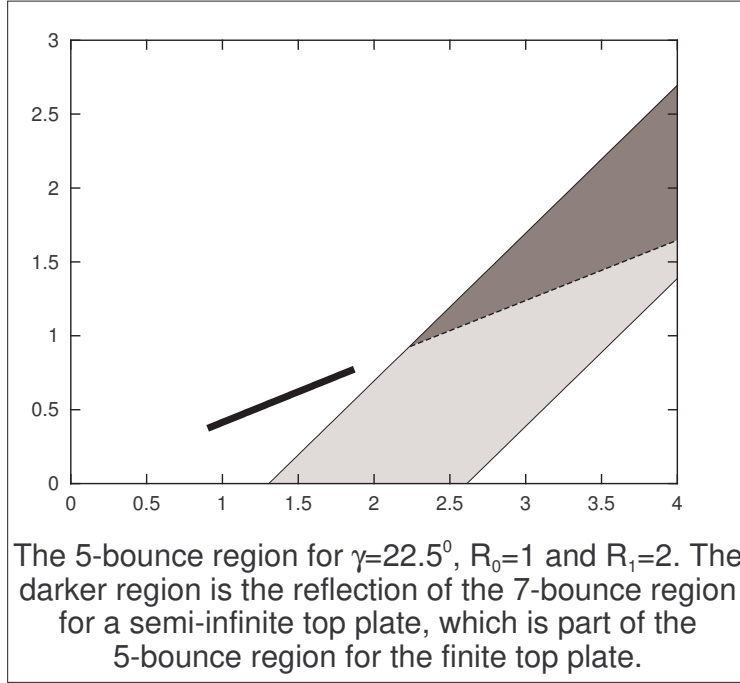
for m_1 odd.

Note:

- (1) If the above restrictions hold for $n = n_0$ then they hold for $n = n_0 - 1$. In other words, the restrictions give an upper bound on the number of contributions of closed bounces.
- (2) As $\beta \rightarrow \pi$ the restrictions reduce to $\frac{2m_0-1}{2m_0}\pi < \beta \leq \frac{2m_0+1}{2m_0+2}\pi$.

5.2. The Odd Bounce Contribution. The situation for odd bounce paths is quite different than for even bounce paths. Due to the finite size of the top plates there are again restrictions on the regions of integration. However, rather than eliminating the contribution from higher bounces, as in the even case, these contributions give lower bounce paths originating from regions outside of the wedge.

In the diagram below the region \mathcal{D}_5 for the closed 5-bounce paths is shown for $\gamma = 22.5^\circ$, $R_0 = 1$ and $R_1 = 2$. This includes a region outside of the wedge (darker shading), which is the reflection of the 7-bounce region that would exist if the top plate were larger.



Since reflection preserves lengths and areas, the total contribution of the odd bounces can be calculated by integrating with only the lower bound on R . That is, we compute the energy for a semi-infinite top plate.

Consider a closed $2m + 1$ -bounce path, with $m = 2n$, which hits the horizontal plate first. As we saw, such a path will exist only if $\beta > \frac{2m-1}{2m}\pi$. The sequence of intersection points (4.21), together with the restriction leads to the lower bound on R :

$$R \geq -\frac{R_0 \cos \beta}{\sin(\psi + 2n\beta)}.$$

On the other hand, we have seen that a closed $4n + 1$ bounce which hits the horizontal plate first exists only if $\psi > 2n(\pi - \beta)$. Thus the limits of integration for ψ are $\psi_0 \leq \psi \leq \frac{\pi}{2}$, where $\psi_0 = \max[2n(\pi - \beta), \beta - \frac{\pi}{2}]$. Thus

$$\begin{aligned} \mathcal{E}_{4n+1}^H &= \frac{\hbar c W}{32\pi^2} \int_{\psi_0}^{\frac{\pi}{2}} \int_{-\frac{R_0 \cos \beta}{\sin(\psi+2n\beta)}}^{\infty} \frac{1}{R^3 \cos^4(\psi + 2n\beta)} dR d\psi \\ &= \frac{\hbar c W}{64\pi^2 \cos^2 \beta R_0^2} \int_{\psi_0}^{\frac{\pi}{2}} \frac{\sin^2(\psi + 2n\beta)}{\cos^4(\psi + 2n\beta)} d\psi \\ &= -\frac{\hbar c W}{192\pi^2 \cos^2 \beta R_0^2} \left[\frac{\sin^3(\psi + 2n\beta)}{\cos^3(\psi + 2n\beta)} \right]_{\psi_0}^{\frac{\pi}{2}} d\psi \\ &= -\frac{\hbar c W}{192\pi^2 \cos^2 \beta R_0^2} (\cot^3(2n\beta) - \epsilon \cot^3[(2n+1)\beta]), \end{aligned}$$

where $\epsilon = 0$ if $2n(\pi - \beta) > \beta - \frac{\pi}{2}$ and $\epsilon = 1$ if $2n(\pi - \beta) \leq \beta - \frac{\pi}{2}$. We denote these two contributions by $\mathcal{E}_{4n+1}^{H\epsilon}$.

Consider the case where $m = 2n - 1$. Again we encounter a restriction on the domain of integration analogous to the previous case. The result is:

$$\mathcal{E}_{4n-1}^{H\epsilon} = -\frac{\hbar c W}{192\pi^2 R_0^2} (\cot^3[(2n-1)\beta] - \epsilon \cot^3(2n\beta)),$$

where $\epsilon = 0$ when $(2n-1)(\pi - \beta) > \beta - \frac{\pi}{2}$ and $\epsilon = 1$ when $(2n-1)(\pi - \beta) < \beta - \frac{\pi}{2}$.

Finally we look at odd bounces that first strike the top plate. The sequence of intersection points is now given by (4.24). The resulting energies are:

$$\mathcal{E}_{4n+1}^{T\epsilon} = -\frac{\hbar c W}{192\pi^2 R_0^2} (\cot^3(2n\beta) - \epsilon \cot^3[(2n+1)\beta]),$$

$$\mathcal{E}_{4n-1}^{T\epsilon} = -\frac{\hbar c W}{192\pi^2 \cos^2 \beta R_0^2} (\cot^3[(2n-1)\beta] - \epsilon \cot^3(2n\beta)),$$

where $\epsilon = 0$ when $m\beta - (2m-1)\pi > \frac{\pi}{2}$ and $\epsilon = 1$ when $m\beta - (2m-1)\pi < \frac{\pi}{2}$.

Summing these four energies we have that the contribution of the odd bounces to the Casimir energy in a wedge with $\frac{2m_0-1}{2m_0}\pi < \beta \leq \frac{2m_0+1}{2m_0+2}\pi$ is

$$\mathcal{E}_{\text{odd}} = \sum_{m=1}^{m_0-1} (\mathcal{E}_{2m+1}^{H1} + \mathcal{E}_{2m+1}^{T1}) + \mathcal{E}_{2m_0+1}^{H0} + \mathcal{E}_{2m_0+1}^{T0} = -\frac{\hbar c W (1 + \cos^2 \beta) \cos \beta}{192\pi^2 R_0^2 \sin^3 \beta}$$

Note that, since this term does not involve R_1 , it is independent of the length of the top plate. From a physical point of view, such contributions to the force have no significance, and are excluded [7]. Thus we only have the even bounce contributions, and Main Theorem 2 follows from Proposition 13 with $\gamma = \pi - \beta$.

6. THE PARALLEL PLATE LIMIT

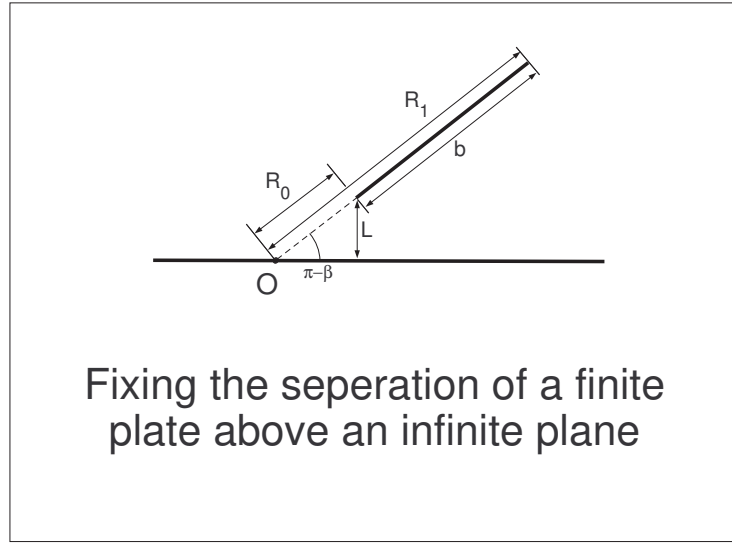
From Casimir's original work [2], the Dirichlet energy between two parallel plates of area A and separation L was computed to be

$$\mathcal{E} = -\frac{\hbar c \pi^2 A}{1440 L^3}.$$

To date, this is the only closed analytic expression for the Casimir energy. In this section we retrieve this result as the limit of our expressions for the energy between non-parallel plates.

Let us consider the limit $\beta \rightarrow \pi$. As it stands, the Casimir energy between a finite plate and an infinite plane, as given above, diverges as $\beta \rightarrow \pi$. This is because the separation between the boundaries goes to zero in this limit.

Before taking the limit, we fix the non-zero minimum separation L between the plates by letting $R_0 \sin \beta = L$ and $b = R_1 - R_0$, where b is the length of the top plate (see the diagram below).



While for each $\beta < \pi$ there is only a finite number of contributions to the Casimir energy, as $\beta \rightarrow \pi$ we pick up an infinite number of terms in the sum. Thus the parallel plate limit needs careful consideration.

Main Theorem 3.

In the parallel plate limit we retrieve Casimir's original result:

$$\lim_{\beta \rightarrow \pi} \mathcal{E} = -\frac{\hbar c \pi^2 b W}{1440 L^3}.$$

Proof. First we interchange the limit with the sum:

$$\lim_{\beta \rightarrow \pi} \mathcal{E} = 2 \sum_{m=1}^{\infty} \lim_{\beta \rightarrow \pi} \mathcal{E}_{2m}.$$

We can do this since for each angle $\beta < \pi$ there are only a finite number of non-zero contributions, and the energy is therefore equicontinuous.

Next, the limit can be computed directly. As noted earlier, the extra restrictions on ψ encountered in computing the even contributions for the finite plate disappear

as $\beta \rightarrow \pi$. Thus, in the limit,

$$\mathcal{E}_{2m} \rightarrow \frac{\hbar c W \cos^2(m\beta) \sin^3 \beta}{64\pi^2 \sin^4(m\beta)} \left(\frac{1}{L^2 \cos \beta} - \frac{1}{(L + b \sin \beta)^2 \cos[(m-1)\beta] \cos(m\beta)} \right).$$

We apply L'Hôpital's four times to show that

$$\lim_{\beta \rightarrow \pi} \frac{\sin^3 \beta ((L + b \sin \beta)^2 \cos[(m-1)\beta] \cos(m\beta) - L^2 \cos \beta)}{\sin^4(2m\beta)} = -\frac{2bL}{m^4}.$$

Thus

$$\lim_{\beta \rightarrow \pi} \mathcal{E}_{2m} = -\frac{\hbar c b W}{32\pi^2 L^3 m^4}$$

Finally the result follows:

$$\lim_{\beta \rightarrow \pi} \mathcal{E} = 2 \sum_{m=1}^{\infty} \lim_{\beta \rightarrow \pi} \mathcal{E}_{2m} = -\frac{\hbar c b W}{16\pi^2 L^3} \sum_{m=1}^{\infty} \frac{1}{m^4} = -\frac{\hbar c \pi^2 b W}{1440 L^3}.$$

□

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