
Preprints

Technology, Engineering & Mathematics

2006-3

Reflection of a Wave off a Surface

Brendan Guilfoyle

Wilhelm Klingenberg

Follow this and additional works at: https://sword.cit.ie/dpttem_kpre



Part of the [Mathematics Commons](#)

REFLECTION OF A WAVE OFF A SURFACE

BRENDAN GUILFOYLE AND WILHELM KLINGENBERG

ABSTRACT. Recent advances in twistor theory are applied to geometric optics in \mathbb{R}^3 . The general formulae for reflection of a wavefront in a surface are derived and in three special cases explicit descriptions are provided: when the reflecting surface is a plane, when the incoming wave is a plane and when the incoming wave is spherical. In each case particular examples are computed exactly and the results plotted to illustrate the outgoing wavefront.

1. INTRODUCTION

In geometric optics, Huygens' principle allows one to describe the propagation of light from two alternative perspectives: one can trace the rays or one can trace the wavefronts. The drawback with the former description is that following a finite number of rays may not describe all of the phenomenon under study, while the formation of caustics in wavefronts can cause difficulties in the latter description [1].

In this paper we utilise recent work [2] in twistor theory to go back and forth between these two perspectives and thus describe the most elementary of optical phenomena: reflection. In particular, we consider the following situation: an incoming wave of light is reflected on a surface in \mathbb{R}^3 . Given the shape of the incoming wavefront and the reflecting surface, can one describe the reflected wavefront? Throughout, we assume that the medium is homogenous and that the speed of propagation is unity.

The technique we employ in answering this question comes from the minitwistor correspondence which identifies the space of oriented affine lines in \mathbb{R}^3 with the tangent bundle to the 2-sphere. This has a long history and has been used in various contexts. In particular, it has been used in the construction of minimal surfaces [8], solutions to the wave equation [9] and the monopole equation [4].

The general context of this work is within the study of line congruences, that is, 2-parameter families of oriented lines in \mathbb{R}^3 [5] [6] [7]. A line congruence is integrable if it is orthogonal to a family of surfaces in \mathbb{R}^3 . At the outset it is not clear that the reflection of an integrable congruence in an arbitrary surface is itself integrable. However, in Theorem 3 that is precisely what we prove. This is the celebrated Theorem of Malus, independently proven by Hamilton in 1827.

In our formalism, going from an integrable line congruence to the orthogonal surface is equivalent to inverting a Cauchy-Riemann operator. The reflection problem can then be broken down into four steps. First we transfer from the incoming wavefront to its associated ray system and we do the same for the reflecting surface. The reflection law relates the outgoing rays to these two line congruences at

the point of contact, and, finally, we transform the outgoing rays to the associated wavefront.

We present the general formulae relating all of these stages for an arbitrary incoming wavefront and reflecting surface (Theorem 2). We go on to give explicit descriptions for three special cases: when the reflecting surface is a plane (Proposition 2), when the incoming wave is a plane (Proposition 3) and when the incoming wave is spherical (Proposition 5). In each case, particular examples are computed exactly and the results plotted to illustrate the outgoing wavefront.

The next section contains the background details of the twistor construction we use - further details can be found in [3]. In section 3 we deduce the required reflection law and establish the fact that the outgoing ray system is integrable if and only if the incoming rays are integrable.

In section 4 we turn to the simplest type of reflection, namely, reflection of an arbitrary wavefront in a plane. Section 5 deals with the general formulae for reflection of a plane wavefront in an arbitrary surface. As an illustration of our technique we determine the reflection of a plane wave in a sphere and a torus. In the final section we give the general formulae for reflection of a spherical wavefront in an arbitrary surface, and illustrate the technique in the case where the surface is the unit sphere.

2. THE TWISTOR CORRESPONDENCE

The key to our approach is the following geometric construction. Let (x^1, x^2, x^3) be Euclidean coordinates on $\mathbb{R}^3 = \mathbb{C} \oplus \mathbb{R}$ and set $z = x^1 + ix^2$, $t = x^3$.

Consider an oriented line in \mathbb{R}^3 passing through (z, t) with direction given by $\xi \in S^2 \subset \mathbb{R}^3$, where ξ is obtained by stereographic projection from the south pole of the unit sphere about the origin onto the plane through the equator.

Then the minimal distance vector from the line to the origin is given by [2]

$$\eta \frac{\partial}{\partial \xi} + \bar{\eta} \frac{\partial}{\partial \bar{\xi}} \in T_\xi S^2,$$

where

$$\eta = \frac{1}{2}(z - 2t\xi - \bar{z}\xi^2),$$

and the point (z, t) is a distance

$$r = \frac{\bar{\xi}z + \xi\bar{z} + (1 - \xi\bar{\xi})t}{1 + \xi\bar{\xi}}.$$

from the point on the line closest to the origin.

Conversely, given η and r and the direction of the line ξ , the point (z, t) in \mathbb{R}^3 can be found by

$$z = \frac{2(\eta - \bar{\eta}\xi^2) + 2\xi(1 + \xi\bar{\xi})r}{(1 + \xi\bar{\xi})^2}, \quad (2.1)$$

$$t = \frac{-2(\eta\bar{\xi} + \bar{\eta}\xi) + (1 - \xi^2\bar{\xi}^2)r}{(1 + \xi\bar{\xi})^2}. \quad (2.2)$$

Now an oriented line in \mathbb{R}^3 is uniquely determined by ξ and η as above. This is the minitwistor correspondence [4], where the space of oriented lines in \mathbb{R}^3 is identified with the tangent bundle to the unit sphere about the origin.

Definition 1. A *line congruence* is a 2-parameter family of oriented lines.

Definition 2. A line congruence is *integrable* if it is orthogonal to a family of surfaces in \mathbb{R}^3 .

Suppose the line congruence is given by $\nu \rightarrow (\xi(\nu, \bar{\nu}), \eta(\nu, \bar{\nu}))$, for $\nu \in \mathbb{C}$.

Theorem 1. [3] *A line congruence is integrable iff*

$$\partial \left(\frac{\eta}{(1 + \xi\bar{\xi})^2} \bar{\partial}\bar{\xi} + \frac{\bar{\eta}}{(1 + \xi\bar{\xi})^2} \bar{\partial}\xi \right) = \bar{\partial} \left(\frac{\bar{\eta}}{(1 + \xi\bar{\xi})^2} \partial\xi + \frac{\eta}{(1 + \xi\bar{\xi})^2} \partial\bar{\xi} \right), \quad (2.3)$$

where ∂ is differentiation with respect to ν .

This equation is the integrability condition for the existence of a real solution r to the following equation

$$\bar{\partial}r = \frac{2(\eta\bar{\partial}\bar{\xi} + \bar{\eta}\bar{\partial}\xi)}{(1 + \xi\bar{\xi})^2}, \quad (2.4)$$

where r is now considered as a function of ν and $\bar{\nu}$.

Thus, by the above theorem, given an integrable line congruence, we can find a local description for the orthogonal surfaces by inverting the $\bar{\partial}$ operator. The real constant of integration gives the affine parameter along the lines normal to these surfaces. An explicit description of the surfaces in \mathbb{R}^3 can then be obtained by inserting ξ , η and r , as functions of ν and $\bar{\nu}$ into (2.1) and (2.2).

Away from flat points, a surface can be parameterized by its normal direction [3], i.e. $\nu = \xi$ and $\eta = F(\xi, \bar{\xi})$ for some complex function F . In this case we refer to F as the *twistor function* of the surface, the integrability condition (2.3) reduces to the simpler

$$\partial \left(\frac{F}{(1 + \xi\bar{\xi})^2} \right) = \bar{\partial} \left(\frac{\bar{F}}{(1 + \xi\bar{\xi})^2} \right),$$

and the function r satisfies

$$\bar{\partial}r = \frac{2F}{(1 + \xi\bar{\xi})^2}.$$

We will refer to r as the *potential function* for the surface.

At a number of points in this paper we use Euclidean motions to simplify the equations for reflection. A translation which takes the origin to (z, t) is the quadratic holomorphic transformation

$$\xi \rightarrow \xi, \quad \eta \rightarrow \eta + \frac{1}{2}(z - 2t\xi - \bar{z}\xi^2),$$

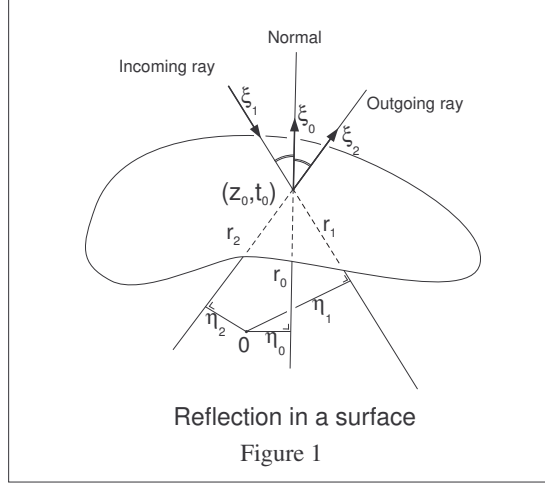
while a rotation about the origin is given by the fractional linear transformation:

$$\xi \rightarrow \frac{\alpha\xi - \bar{\beta}}{\beta\xi + \bar{\alpha}}, \quad \eta \rightarrow \frac{\eta}{(\beta\xi + \bar{\alpha})^2},$$

where $\alpha, \beta \in \mathbb{C}$ satisfy $\alpha\bar{\alpha} + \beta\bar{\beta} = 1$.

3. REFLECTION IN A SURFACE

Consider a ray with direction ξ_1 and perpendicular distance vector η_1 striking a surface S at the point (z_0, t_0) , where the normal direction is ξ_0 and the perpendicular distance vector is η_0 . Let ξ_2 be the direction of the outgoing wave and η_2 be the perpendicular distance vector (see Figure 1).



For the surface S , suppose that the point of reflection is a distance r_0 from the minimal distance point on the normal line to the origin, so that, by (2.1) and (2.2)

$$z_0 = \frac{2(\eta_0 - \bar{\eta}_0 \xi_0^2) + 2\xi_0(1 + \xi_0 \bar{\xi}_0)r_0}{(1 + \xi_0 \bar{\xi}_0)^2}, \quad t_0 = \frac{-2(\eta_0 \bar{\xi}_0 + \bar{\eta}_0 \xi_0) + (1 - \xi_0^2 \bar{\xi}_0^2)r_0}{(1 + \xi_0 \bar{\xi}_0)^2}.$$

On the other hand, since both the incoming and outgoing rays contain (z_0, t_0) ,

$$\eta_i = \frac{1}{2}(z_0 - 2t_0 \xi_i - \bar{z}_0 \xi_i^2),$$

for $i = 1, 2$.

Combining the previous three equations we have, after some rearrangement,

$$\eta_i = \frac{(1 + \bar{\xi}_0 \xi_i)^2 \eta_0 - (\xi_0 - \xi_i)^2 \bar{\eta}_0 + (\xi_0 - \xi_i)(1 + \bar{\xi}_0 \xi_i)(1 + \xi_0 \bar{\xi}_0)r_0}{(1 + \xi_0 \bar{\xi}_0)^2}. \quad (3.1)$$

We turn now to the law of reflection.

Proposition 1. *If ξ_1 is the direction of the incoming ray, ξ_2 the outgoing ray direction and ξ_0 the normal direction at the point of reflection, then*

$$\xi_2 = \frac{2\xi_0 \bar{\xi}_1 + 1 - \xi_0 \bar{\xi}_0}{(1 - \xi_0 \bar{\xi}_0) \bar{\xi}_1 - 2\xi_0}. \quad (3.2)$$

Proof. A rotation about any point ξ_0 on \mathbb{P}^1 is described by a unitary fractional linear transformation:

$$\text{Rot}_{\xi_0}(\xi) = \frac{\alpha \xi - \bar{\beta}}{\beta \xi + \bar{\alpha}},$$

where $\alpha, \beta \in \mathbb{C}$ satisfy $\alpha \bar{\alpha} + \beta \bar{\beta} = 1$. The inverse rotation is given by

$$\text{Rot}_{\xi_0}^{-1}(\xi) = \frac{\bar{\alpha} \xi + \bar{\beta}}{-\beta \xi + \alpha}. \quad (3.3)$$

Let S_{ξ_0} be the rotation

$$S_{\xi_0}(\xi) = \frac{\alpha \xi - \bar{\beta}}{\beta \xi + \bar{\alpha}},$$

where

$$\alpha = \frac{1}{\sqrt{1 + \xi_0 \bar{\xi}_0}} \quad \beta = \frac{\bar{\xi}_0}{\sqrt{1 + \xi_0 \bar{\xi}_0}}.$$

Clearly S_{ξ_0} is a rotation that takes ξ_0 to zero (the North pole). Thus, if we denote reflection through ξ_0 by R_{ξ_0} , then $R_{\xi_0} = S_{\xi_0}^{-1} \circ R_0 \circ S_{\xi_0}$.

Now $R_0(\xi) = -\xi$, so using (3.3) and the definition of S_{ξ_0} we find that

$$R_{\xi_0}(\xi) = \frac{(\xi_0 \bar{\xi}_0 - 1)\xi + 2\xi_0}{2\bar{\xi}_0 \xi + 1 - \xi_0 \bar{\xi}_0}.$$

Alternatively, we can define R_{ξ_0} as above and check that it is a rotation (obvious), and satisfies $R_{\xi_0}(\xi_0) = \xi_0$ and $R_{\xi_0} \circ R_{\xi_0}(\xi) = \xi$. Thus it is a rotation through 180° about ξ_0 i.e. reflection in ξ_0 .

Finally, the law of reflection says that the direction ξ_2 of the outgoing ray is obtained by reflecting the antipodal direction of the incoming ray ξ_1 through the normal direction ξ_0 at the point of reflection.

To complete the proposition, we note that the antipodal map on \mathbb{P}^1 is $\xi \rightarrow -\bar{\xi}^{-1}$, so that

$$\xi_2 = R_{\xi_0} \left(-\frac{1}{\bar{\xi}_1} \right) = \frac{2\xi_0 \bar{\xi}_1 + 1 - \xi_0 \bar{\xi}_0}{(1 - \xi_0 \bar{\xi}_0) \bar{\xi}_1 - 2\bar{\xi}_0},$$

as claimed. \square

The equations governing reflection are:

Theorem 2. *Consider a wavefront given by $\nu_1 \rightarrow (\xi_1(\nu_1, \bar{\nu}_1), \eta_1(\nu_1, \bar{\nu}_1))$ reflecting off a surface given by $\nu_0 \rightarrow (\xi_0(\nu_0, \bar{\nu}_0), \eta_0(\nu_0, \bar{\nu}_0))$ and $r_0(\nu_0, \bar{\nu}_0)$. Then the reflected wavefront is determined by*

$$\xi_2 = \frac{2\xi_0 \bar{\xi}_1 + 1 - \xi_0 \bar{\xi}_0}{(1 - \xi_0 \bar{\xi}_0) \bar{\xi}_1 - 2\bar{\xi}_0}, \quad (3.4)$$

$$\eta_1 = \frac{(1 + \bar{\xi}_0 \xi_1)^2 \eta_0 - (\xi_0 - \xi_1)^2 \bar{\eta}_0 + (\xi_0 - \xi_1)(1 + \bar{\xi}_0 \xi_1)(1 + \xi_0 \bar{\xi}_0) r_0}{(1 + \xi_0 \bar{\xi}_0)^2} \quad (3.5)$$

$$\eta_2 = \frac{(\bar{\xi}_0 - \bar{\xi}_1)^2 \eta_0 - (1 + \xi_0 \bar{\xi}_1)^2 \bar{\eta}_0 + (\bar{\xi}_0 - \bar{\xi}_1)(1 + \xi_0 \bar{\xi}_1)(1 + \xi_0 \bar{\xi}_0) r_0}{((1 - \xi_0 \bar{\xi}_0) \bar{\xi}_1 - 2\bar{\xi}_0)^2}. \quad (3.6)$$

Equation (3.5) determines the intersection of the incident rays with the surface, while (3.4) and (3.6) determine the direction and perpendicular distance from the origin of the reflected rays.

Proof. These come from combining the reflection law (3.2) with (3.1), after some rearrangement for η_2 . \square

In general, the reflected line congruence can be parameterized by ν_1 , the parameter for the incoming wavefront.

Theorem 3. *The surface, incident and reflected congruences satisfy the following relationship:*

$$\frac{\eta_2 \bar{\partial}_1 \bar{\xi}_2 + \bar{\eta}_2 \bar{\partial}_1 \xi_2}{(1 + \xi_2 \bar{\xi}_2)^2} = \frac{\eta_1 \bar{\partial}_1 \bar{\xi}_1 + \bar{\eta}_1 \bar{\partial}_1 \xi_1}{(1 + \xi_1 \bar{\xi}_1)^2} + \bar{\partial}_1 \left(\frac{|\xi_0 - \xi_1|^2 - |1 + \xi_0 \bar{\xi}_1|^2}{(1 + \xi_0 \bar{\xi}_0)(1 + \xi_1 \bar{\xi}_1)} r_0 \right), \quad (3.7)$$

where ∂_1 is differentiation with respect to ν_1 , the parameter for the incoming wave.

Proof. We start by differentiating the reflection law (3.4):

$$\begin{aligned}\partial_1 \xi_2 &= \frac{-(1 + \xi_0 \bar{\xi}_0)^2 \partial_1 \bar{\xi}_1 + 2(\bar{\xi}_0 - \bar{\xi}_1)^2 \partial_1 \xi_0 + 2(1 + \xi_0 \bar{\xi}_1)^2 \partial_1 \bar{\xi}_0}{[(1 - \xi_0 \bar{\xi}_0) \bar{\xi}_1 - 2\bar{\xi}_0]^2}, \\ \bar{\partial}_1 \xi_2 &= \frac{-(1 + \xi_0 \bar{\xi}_0)^2 \bar{\partial}_1 \bar{\xi}_1 + 2(\bar{\xi}_0 - \bar{\xi}_1)^2 \bar{\partial}_1 \xi_0 + 2(1 + \xi_0 \bar{\xi}_1)^2 \bar{\partial}_1 \bar{\xi}_0}{[(1 - \xi_0 \bar{\xi}_0) \bar{\xi}_1 - 2\bar{\xi}_0]^2}.\end{aligned}$$

The reflection law (3.4) also implies

$$1 + \xi_2 \bar{\xi}_2 = \frac{(1 + \xi_0 \bar{\xi}_0)^2 (1 + \xi_1 \bar{\xi}_1)}{[(1 - \xi_0 \bar{\xi}_0) \bar{\xi}_1 - 2\bar{\xi}_0][(1 - \xi_0 \bar{\xi}_0) \bar{\xi}_1 - 2\bar{\xi}_0]}.$$

Thus, using (3.5) and (3.6) we find

$$\begin{aligned}\frac{\eta_2 \bar{\partial}_1 \bar{\xi}_2 + \bar{\eta}_2 \bar{\partial}_1 \xi_2}{(1 + \xi_2 \bar{\xi}_2)^2} &= \frac{\eta_1 \bar{\partial}_1 \bar{\xi}_1 + \bar{\eta}_1 \bar{\partial}_1 \xi_1}{(1 + \xi_1 \bar{\xi}_1)^2} + \frac{2(|\xi_0 - \xi_1|^2 - |1 + \xi_0 \bar{\xi}_1|^2)(\eta_0 \bar{\partial}_1 \bar{\xi}_0 + \bar{\eta}_0 \bar{\partial}_1 \xi_0)}{(1 + \xi_0 \bar{\xi}_0)^3 (1 + \xi_1 \bar{\xi}_1)} \\ &\quad + \frac{2[(\xi_0 - \xi_1)(1 + \xi_0 \bar{\xi}_1)][(1 + \xi_1 \bar{\xi}_1) \bar{\partial}_1 \bar{\xi}_0 - (1 + \xi_0 \bar{\xi}_0) \bar{\partial}_1 \bar{\xi}_1]}{(1 + \xi_0 \bar{\xi}_0)^2 (1 + \xi_1 \bar{\xi}_2)} r_0 \\ &\quad + \frac{2[(\bar{\xi}_0 - \bar{\xi}_1)(1 + \bar{\xi}_0 \xi_1)][(1 + \xi_1 \bar{\xi}_1) \bar{\partial}_1 \xi_0 - (1 + \xi_0 \bar{\xi}_0) \bar{\partial}_1 \xi_1]}{(1 + \xi_0 \bar{\xi}_0)^2 (1 + \xi_1 \bar{\xi}_2)} r_0.\end{aligned}\tag{3.8}$$

Now, since we are reflecting in a surface, we have the potential function r_0 satisfying

$$\frac{2(\eta_0 \bar{\partial}_0 \bar{\xi}_0 + \bar{\eta}_0 \bar{\partial}_0 \xi_0)}{(1 + \xi_0 \bar{\xi}_0)^2} = \bar{\partial}_0 r_0,$$

and so

$$\begin{aligned}\frac{2(\eta_0 \bar{\partial}_1 \bar{\xi}_0 + \bar{\eta}_0 \bar{\partial}_1 \xi_0)}{(1 + \xi_0 \bar{\xi}_0)^2} &= \frac{2(\eta_0 (\bar{\partial}_1 \bar{\nu}_0 \bar{\partial}_0 \bar{\xi}_0 + \bar{\partial}_1 \nu_0 \partial_0 \bar{\xi}_0) + \bar{\eta}_0 (\bar{\partial}_1 \nu_0 \partial_0 \xi_0 + \bar{\partial}_1 \bar{\nu}_0 \bar{\partial}_0 \xi_0))}{(1 + \xi_0 \bar{\xi}_0)^2} \\ &= \frac{2(\bar{\partial}_1 \bar{\nu}_0 (\eta_0 \bar{\partial}_0 \bar{\xi}_0 + \bar{\eta}_0 \bar{\partial}_0 \xi_0) + \bar{\partial}_1 \nu_0 (\eta_0 \partial_0 \bar{\xi}_0 + \bar{\eta}_0 \partial_0 \xi_0))}{(1 + \xi_0 \bar{\xi}_0)^2} \\ &= \bar{\partial}_1 \bar{\nu}_0 \bar{\partial}_0 r_0 + \bar{\partial}_1 \nu_0 \partial_0 r_0 \\ &= \bar{\partial}_1 r_0.\end{aligned}$$

Substituting this in (3.8) we get that

$$\begin{aligned}\frac{\eta_2 \bar{\partial}_1 \bar{\xi}_2 + \bar{\eta}_2 \bar{\partial}_1 \xi_2}{(1 + \xi_2 \bar{\xi}_2)^2} &= \frac{\eta_1 \bar{\partial}_1 \bar{\xi}_1 + \bar{\eta}_1 \bar{\partial}_1 \xi_1}{(1 + \xi_1 \bar{\xi}_1)^2} + \frac{|\xi_0 - \xi_1|^2 - |1 + \xi_0 \bar{\xi}_1|^2}{(1 + \xi_0 \bar{\xi}_0)(1 + \xi_1 \bar{\xi}_1)} \bar{\partial}_1 r_0 \\ &\quad + \frac{2[(\xi_0 - \xi_1)(1 + \xi_0 \bar{\xi}_1)][(1 + \xi_1 \bar{\xi}_1) \bar{\partial}_1 \bar{\xi}_0 - (1 + \xi_0 \bar{\xi}_0) \bar{\partial}_1 \bar{\xi}_1]}{(1 + \xi_0 \bar{\xi}_0)^2 (1 + \xi_1 \bar{\xi}_2)} r_0 \\ &\quad + \frac{2[(\bar{\xi}_0 - \bar{\xi}_1)(1 + \bar{\xi}_0 \xi_1)][(1 + \xi_1 \bar{\xi}_1) \bar{\partial}_1 \xi_0 - (1 + \xi_0 \bar{\xi}_0) \bar{\partial}_1 \xi_1]}{(1 + \xi_0 \bar{\xi}_0)^2 (1 + \xi_1 \bar{\xi}_2)} r_0,\end{aligned}$$

and the last three terms on the right hand side are $\bar{\partial}_1$ of the real function, as stated in the theorem. \square

As a corollary we get the Theorem of Malus:

Corollary 1. *A reflected congruence is integrable if and only if the initial congruence is integrable.*

Proof. Since the second term on the right-hand side of (3.7) is $\bar{\partial}_1$ of a real function, the integrability condition (2.3) means that the outgoing congruence is integrable if and only if the incoming wave is integrable. \square

In the next sections we consider three special cases: that of reflection of an arbitrary wavefront in a plane, and that of plane and spherical wavefronts reflected off an arbitrary surface. In all cases the resulting wavefront can be explicitly determined.

More generally, assume that the incident wave can be described by $\eta_1 = F_1(\xi_1, \bar{\xi}_1)$ for some complex function F_1 . Suppose further that we can invert (3.5) for ξ_1 as a function of ξ_0 and η_0 . Then we can substitute this in (3.4) and (3.6) to find both ξ_2 and η_2 as a function of ξ_0 and η_0 . Finally, we can find the potential function r_2 by integrating (2.4).

4. REFLECTION IN A PLANE

Consider an arbitrary wave described parametrically by $\xi_1(\nu, \bar{\nu})$ and $\eta_1(\nu, \bar{\nu})$. We want to determine the resulting wave after reflection in a plane. By a rotation we can align the plane with the x^1x^2 -plane so that the initial wave lies in the region $x^3 > 0$.

Proposition 2. *The reflection of a wave given by $\xi_1(\nu, \bar{\nu})$ and $\eta_1(\nu, \bar{\nu})$ in the x^1x^2 -plane is*

$$\xi_2 = \frac{1}{\xi_1}, \quad \eta_2 = -\frac{\bar{\eta}_1}{\xi_1^2}. \quad (4.1)$$

Proof. The x^1x^2 -plane is given by $\xi_0 = r_0 = 0$. Inserting this in the reflection law (3.4) immediately gives the first of the above equations. Inserting it in (3.5) and (3.6) we get

$$\eta_1 = \eta_0 - \xi_1^2 \bar{\eta}_0, \quad \eta_2 = \eta_0 - \frac{\bar{\eta}_0}{\xi_1^2}. \quad (4.2)$$

Adding the first of these to ξ_1^2 times its complex conjugate gives

$$\eta_0 = \frac{\eta_1 + \xi_1^2 \bar{\eta}_1}{1 - \xi_1^2 \bar{\xi}_1^2}.$$

Inserting this in the second of (4.2) gives the result. \square

4.1. Reflection of a Spherical Wavefront in a Plane. Consider a spherical wavefront with source $(0, 0, t_1)$. This congruence is given by $\eta_1 = -2t_1\xi_1$. Substituting this in (4.1) we find

$$\xi_2 = \frac{1}{\xi_1} \quad \eta_2 = \frac{2t_1}{\xi_1}.$$

Combining these two we see that $\eta_2 = 2t_1\xi_2$, which is a spherical congruence with source $(0, 0, -t_1)$. Thus we retrieve the well-known law of reflection, whereby the reflection of a spherical wave centered at $(0, 0, t_1)$ in a plane mirror is another spherical wave with virtual centre $(0, 0, -t_1)$.

5. REFLECTION OF A PLANE WAVEFRONT

Consider an incoming wavefront with fixed direction, i.e. a plane wavefront. In this case, the reflected wave can be expressed explicitly in terms of the reflecting surface:

Proposition 3. *The reflection of a plane wavefront with direction ξ_1 off a surface given by $\xi_0(\nu, \bar{\nu})$ and $\eta_0(\nu, \bar{\nu})$ is given by*

$$\xi_2 = \frac{2\xi_0\bar{\xi}_1 + 1 - \xi_0\bar{\xi}_0}{(1 - \xi_0\bar{\xi}_0)\bar{\xi}_1 - 2\xi_0}, \quad (5.1)$$

$$\eta_2 = \frac{(\bar{\xi}_0 - \bar{\xi}_1)^2\eta_0 - (1 + \xi_0\bar{\xi}_1)^2\bar{\eta}_0 + (\bar{\xi}_0 - \bar{\xi}_1)(1 + \xi_0\bar{\xi}_1)(1 + \xi_0\bar{\xi}_0)r_0}{((1 - \xi_0\bar{\xi}_0)\bar{\xi}_1 - 2\xi_0)^2}, \quad (5.2)$$

the reflected congruence being parameterized by the parameter value ν of the surface at the point of reflection.

Proof. Since the incident angle is constant, there is no need to determine the points of intersection, and the above equations are just the equation of reflection (3.4) and equation (3.6). \square

In this proposition, the resulting wavefront is parameterized by the point of reflection on the surface. For a plane wave it is also possible to parameterise the outgoing wave by its direction:

Proposition 4. *Suppose S a surface in \mathbb{R}^3 , given by the $\eta_0 = F_0(\xi, \bar{\xi})$ with potential function $r_0(\xi, \bar{\xi})$, is struck by a plane wave with normal direction $-\bar{\xi}_1^{-1}$. Then the reflected wave is given by the line congruence $(\xi, \eta_2 = F_2(\xi, \bar{\xi}))$ with*

$$F_2(\xi, \bar{\xi}) = \frac{1}{4} \left((1 + \gamma)^2 F_0(\xi_0, \bar{\xi}_0) - (\xi - \gamma\xi_1)^2 \bar{F}_0(\xi_0, \bar{\xi}_0) + 2(\xi_1 - \xi)\gamma r_0(\xi_0, \bar{\xi}_0) \right), \quad (5.3)$$

where

$$\xi_0 = \frac{\xi\bar{\xi}\bar{\xi}_1\bar{\xi}_1 - 1 \pm \sqrt{(1 + \xi\bar{\xi})(1 + \xi_1\bar{\xi}_1)(1 + \xi\bar{\xi}_1)(1 + \xi\bar{\xi}_1)}}{\bar{\xi}(1 + \xi\bar{\xi}_1) + \xi_1(1 + \xi\bar{\xi}_1)}, \quad (5.4)$$

and

$$\gamma = \pm \sqrt{\frac{(1 + \xi\bar{\xi}_1)(1 + \xi\bar{\xi})}{(1 + \xi\bar{\xi}_1)(1 + \xi_1\bar{\xi}_1)}}.$$

The two signs yield the same line congruence with opposite orientation, and can be chosen to point in the outgoing direction.

Proof. From the reflection law (3.4) with $\xi_2 = \xi$ and $\xi_1 \rightarrow -\bar{\xi}_1^{-1}$ we have

$$\xi = \frac{(\xi_0\bar{\xi}_0 - 1)\xi_1 + 2\xi_0}{2\bar{\xi}_0\xi_1 + 1 - \xi_0\bar{\xi}_0}.$$

This can be recast as a quadratic equation for ξ_0 , which has solution (5.4). From these, after some calculation, we find the following expressions

$$\begin{aligned} 1 + \xi_0\bar{\xi}_0 &= \pm \frac{2\sqrt{b^2 + \beta\bar{\beta}}(-b \pm \sqrt{b^2 + \beta\bar{\beta}})}{\beta\bar{\beta}}, & 1 - \xi_0\bar{\xi}_0 &= \frac{2b(-b \pm \sqrt{b^2 + \beta\bar{\beta}})}{\beta\bar{\beta}}, \\ 1 + \xi\bar{\xi}_0 &= \frac{\beta - b\xi \pm \xi\sqrt{b^2 + \beta\bar{\beta}}}{\beta}, & \xi_0 - \xi &= \frac{-\xi\bar{\beta} - b \pm \sqrt{b^2 + \beta\bar{\beta}}}{\bar{\beta}}, \end{aligned}$$

where we have introduced

$$b = 1 - \xi\bar{\xi}\xi_1\bar{\xi}_1 \quad \beta = \xi(1 + \bar{\xi}\xi_1) + \xi_1(1 + \xi\bar{\xi}_1).$$

Then we compute the following combinations of the above

$$\begin{aligned} \frac{1 + \xi\bar{\xi}_0}{1 + \xi_0\bar{\xi}_0} &= \frac{\pm\sqrt{(1 + \xi\bar{\xi}_1)(1 + \xi\bar{\xi})} + \sqrt{(1 + \bar{\xi}\xi_1)(1 + \xi_1\bar{\xi}_1)}}{2\sqrt{(1 + \bar{\xi}\xi_1)(1 + \xi_1\bar{\xi}_1)}}, \\ \frac{\xi_0 - \xi}{1 + \xi_0\bar{\xi}_0} &= \frac{\pm\xi_1\sqrt{(1 + \bar{\xi}\xi_1)(1 + \xi\bar{\xi})} - \xi\sqrt{(1 + \bar{\xi}\xi_1)(1 + \xi_1\bar{\xi}_1)}}{2\sqrt{(1 + \bar{\xi}\xi_1)(1 + \xi_1\bar{\xi}_1)}}, \\ \frac{(\xi_0 - \xi)(1 + \xi\bar{\xi}_0)}{1 + \xi_0\bar{\xi}_0} &= \pm\frac{1}{2}(\xi_1 - \xi)\sqrt{\frac{(1 + \xi\bar{\xi}_1)(1 + \xi\bar{\xi})}{(1 + \bar{\xi}\xi_1)(1 + \xi_1\bar{\xi}_1)}}. \end{aligned}$$

Substituting these in (3.6) gives the result stated. \square

We also have the following corollary for waves travelling down the x^3 -axis:

Corollary 2. *Suppose S a surface in \mathbb{R}^3 , given by the $\eta_0 = F_0(\xi, \bar{\xi})$ with potential function $r_0(\xi, \bar{\xi})$, is struck by a plane wave moving down the x^3 -axis. Then the reflected wave is given by the line congruence $(\xi, \eta_2 = F_2(\xi, \bar{\xi}))$ with*

$$F_2(\xi, \bar{\xi}) = \frac{1}{4} \left(\left(1 + \sqrt{1 + \xi\bar{\xi}} \right)^2 F_0(\xi_0, \bar{\xi}_0) - \xi^2 \bar{F}_0(\xi_0, \bar{\xi}_0) - 2\xi\sqrt{1 + \xi\bar{\xi}} r_0(\xi_0, \bar{\xi}_0) \right), \quad (5.5)$$

where

$$\xi_0 = \frac{-1 + \sqrt{1 + \xi\bar{\xi}}}{\xi}.$$

Proof. This follows from the above Proposition 4 by setting $\xi_1 = 0$. \square

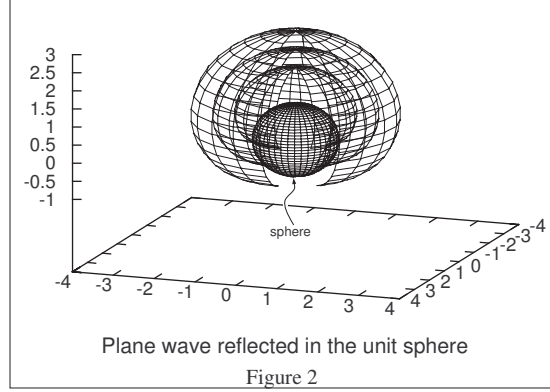
By way of example, we now compute the twistor function for a plane wave after reflection off the unit sphere and the rotationally symmetric torus.

5.1. Reflection of a Plane Wavefront in the Unit Sphere. By translation we can move the unit sphere to the origin in \mathbb{R}^3 and by a rotation we can set the direction of the plane wave to be down the x^3 -axis. Thus we can use corollary 2.

The twistor function and potential for the unit sphere at the origin are $F_0 = 0$ and $r_0 = 1$. Thus, substituting these in (5.5), the reflected wavefront is given by

$$F_2 = -\frac{1}{2}\xi\sqrt{1 + \xi\bar{\xi}}, \quad r_2 = \frac{2}{\sqrt{1 + \xi\bar{\xi}}} + C.$$

Figure 2 shows the resulting outgoing wavefronts (which are axially symmetric, but not spherically symmetric).



5.2. Reflection of a Plane Wavefront in a Torus. Consider the torus with core radius a and meridian radius b which is rotationally symmetric about the x^3 -axis. This has twistor function and potential

$$F_0 = \frac{a}{2}(1 - \xi\bar{\xi})\sqrt{\frac{\xi}{\bar{\xi}}}, \quad r_0 = \frac{2a\sqrt{\xi\bar{\xi}}}{1 + \xi\bar{\xi}} + b.$$

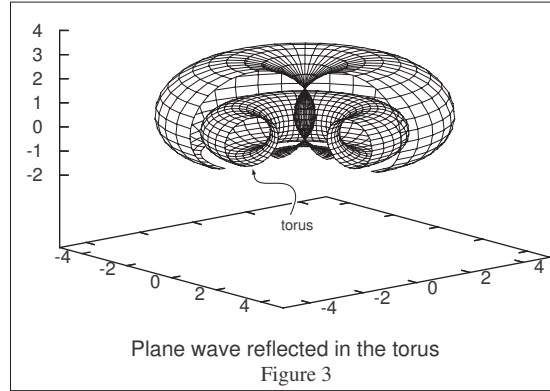
In what follows we will consider the case where $a = 2$ and $b = 1$. We will deal with three cases: a wave travelling down the x^3 -axis, up the x^1 -axis and one striking the torus at an angle of 45° to the vertical.

5.2.1. *Plane wavefront moving down the x^3 -axis.* The line congruence resulting from reflection of a plane wave moving down the x^3 -axis can be found directly from corollary 2. The result is

$$F_2 = \frac{2(1 - \xi\bar{\xi}) - \sqrt{\xi\bar{\xi}}\sqrt{1 + \xi\bar{\xi}}}{2}\sqrt{\frac{\xi}{\bar{\xi}}}, \quad r_2 = \frac{4\sqrt{\xi\bar{\xi}} + 2\sqrt{1 + \xi\bar{\xi}}}{1 + \xi\bar{\xi}},$$

where we have integrated F_2 to obtain r_2 .

Figure 3 shows the complete wavefront as it leaves the torus. Notice the annular shadow.



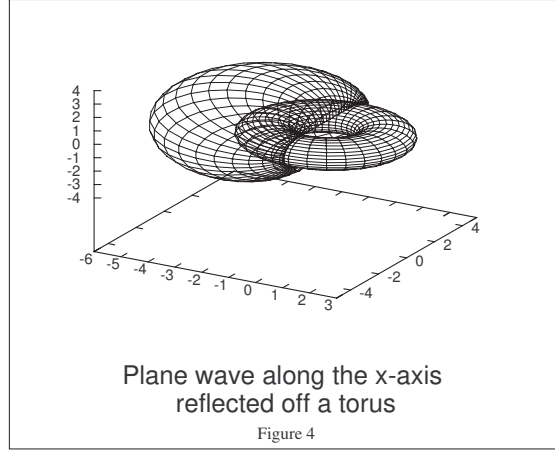
5.2.2. *Plane wavefront moving up the x^1 -axis.* In order to determine the reflection of a plane wave approaching from another direction we shall use Proposition 3. If the direction of the incoming wave is ξ_1 , substitution of the torus twistor function and potential in (5.2) gives

$$\eta_2 = \frac{[(\bar{\xi}_0 - \bar{\xi}_1)^2 \xi_0 - (1 + \xi_0 \bar{\xi}_1)^2 \bar{\xi}_0](1 - \xi_0 \bar{\xi}_0) + (\bar{\xi}_0 - \bar{\xi}_1)(1 + \xi_0 \bar{\xi}_1)[(1 + \xi_0 \bar{\xi}_0)\sqrt{\xi_0 \bar{\xi}_0} + 4\xi_0 \bar{\xi}_0]}{\sqrt{\xi_0 \bar{\xi}_0}((1 - \xi_0 \bar{\xi}_0)\bar{\xi}_1 - 2\bar{\xi}_0)^2} \quad (5.6)$$

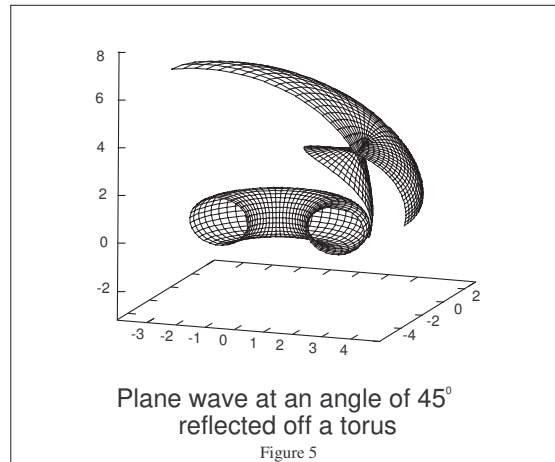
Integrating this gives the potential function

$$r_2 = \frac{4[2(|\bar{\xi}_0 - \bar{\xi}_1|^2 - |1 + \xi_0 \bar{\xi}_1|^2)\sqrt{\xi_0 \bar{\xi}_0} + (\xi_0 \bar{\xi}_0(1 - \xi_1 \bar{\xi}_1) - \bar{\xi}_0 \xi_1 - \xi_0 \bar{\xi}_1)(1 + \xi_0 \bar{\xi}_0)]}{(1 + \xi_1 \bar{\xi}_1)(1 + \xi_0 \bar{\xi}_0)^2} \quad (5.7)$$

For a wave travelling along the x^1 -axis we set $\xi_1 = 1$. The resulting wavefront is shown in Figure 4.



5.2.3. *Plane wavefront at an angle of 45° to the vertical.* Finally, a portion of the reflected wavefront generated by an incident wave making an angle of 45° with the x^3 -axis is shown in Figure 5. This is obtained by setting $\xi_1 = 2.4$ in (5.6) and (5.7).



6. REFLECTION OF A SPHERICAL WAVEFRONT

Assume that the wavefront is spherical, that is, the rays have a single focus. By a translation we can move this focus to the origin and then the wavefront is simply given by $\eta_1 = 0$. Every point in \mathbb{R}^3 , other than the origin, is contained on two oriented lines in the congruence (the same line with both orientations).

Proposition 5. *A spherical wavefront, with focus the origin, reflected off a surface (not containing the origin) determined by $\xi_0(\nu, \bar{\nu})$, $\eta_0(\nu, \bar{\nu})$ and $r_0(\nu, \bar{\nu})$ gives rise to the congruence*

$$\xi_2 = \frac{2\eta_0 + 2\bar{\eta}_0\xi_0^2 \pm 2\xi_0\beta_0}{2(\xi_0\bar{\eta}_0 - \bar{\xi}_0\eta_0) - (1 + \xi_0\bar{\xi}_0)^2 r_0 \pm (1 - \xi_0\bar{\xi}_0)\beta_0}, \quad (6.1)$$

$$\eta_2 = \alpha_1^2\eta_0 - \alpha_2^2\bar{\eta}_0 - 2(1 + \xi_0\bar{\xi}_0)\alpha_3 r_0, \quad (6.2)$$

where

$$\alpha_1 = \frac{2\bar{\eta}_0\xi_0 - (1 + \xi_0\bar{\xi}_0)r_0 \pm \beta_0}{2(\xi_0\bar{\eta}_0 - \bar{\xi}_0\eta_0) - (1 + \xi_0\bar{\xi}_0)^2 r_0 \pm (1 - \xi_0\bar{\xi}_0)\beta_0},$$

$$\alpha_2 = \frac{2\eta_0 + \xi_0(1 + \xi_0\bar{\xi}_0)r_0 \pm \xi_0\beta_0}{2(\xi_0\bar{\eta}_0 - \bar{\xi}_0\eta_0) - (1 + \xi_0\bar{\xi}_0)^2 r_0 \pm (1 - \xi_0\bar{\xi}_0)\beta_0},$$

$$\alpha_3 = \frac{4\eta_0\bar{\eta}_0\xi_0 - (\eta_0 - \bar{\eta}_0\xi_0^2)(1 + \xi_0\bar{\xi}_0)r_0 \pm (\eta_0 + \bar{\eta}_0\xi_0^2)\beta_0}{(2(\xi_0\bar{\eta}_0 - \bar{\xi}_0\eta_0) - (1 + \xi_0\bar{\xi}_0)^2 r_0 \pm (1 - \xi_0\bar{\xi}_0)\beta_0)^2},$$

and

$$\beta_0 = \sqrt{4\eta_0\bar{\eta}_0 + (1 + \xi_0\bar{\xi}_0)^2 r_0^2}.$$

Here the two different signs give the same line congruence with the opposite orientation, and can be chosen to point in the outgoing direction.

Proof. Setting $\eta_1 = 0$ in (3.5) we get a quadratic equation for ξ_1 , with solution:

$$\xi_1 = \frac{2(\eta_0\bar{\xi}_0 + \bar{\eta}_0\xi_0) - (1 - \xi_0^2\bar{\xi}_0^2)r_0 \pm (1 + \xi_0\bar{\xi}_0)\beta_0}{2(\bar{\eta}_0 - \eta_0\xi_0^2) + 2\xi_0(1 + \xi_0\bar{\xi}_0)r_0}.$$

Inserting this in (3.4) yields the expression for ξ_2 in the theorem. In addition we find that

$$\bar{\xi}_1 - \bar{\xi}_0 = \frac{(2\bar{\eta}_0\xi_0 - (1 + \xi_0\bar{\xi}_0)r_0 \pm \beta_0)(1 + \xi_0\bar{\xi}_0)}{2(\eta_0 - \bar{\eta}_0\xi_0^2) + 2\xi_0(1 + \xi_0\bar{\xi}_0)r_0},$$

$$1 + \xi_0\bar{\xi}_1 = \frac{(2\eta_0 + \xi_0(1 + \xi_0\bar{\xi}_0)r_0 \pm \xi_0\beta_0)(1 + \xi_0\bar{\xi}_0)}{2(\eta_0 - \bar{\eta}_0\xi_0^2) + 2\xi_0(1 + \xi_0\bar{\xi}_0)r_0},$$

while

$$(1 - \xi_0\bar{\xi}_0)\bar{\xi}_1 - 2\bar{\xi}_0 = \frac{(2(\xi_0\bar{\eta}_0 - \bar{\xi}_0\eta_0) - (1 + \xi_0\bar{\xi}_0)^2 r_0 \pm (1 - \xi_0\bar{\xi}_0)\beta_0)(1 + \xi_0\bar{\xi}_0)}{2(\eta_0 - \bar{\eta}_0\xi_0^2) + 2\xi_0(1 + \xi_0\bar{\xi}_0)r_0}.$$

Substituting these in (3.6) completes the theorem. \square

6.1. Reflection of a Spherical Wavefront in the Unit Sphere. Consider the unit sphere, centred at $(0,0,-2)$. This is determined by

$$\eta_0 = 2\xi_0, \quad r_0 = 1 - 2\frac{1 - \xi_0\bar{\xi}_0}{1 + \xi_0\bar{\xi}_0}.$$

Thus

$$\beta_0 = \sqrt{1 + 10\xi_0\bar{\xi}_0 + r\xi_0^2\bar{\xi}_0^2},$$

and by (6.1)

$$\xi_2 = \frac{2\xi_0(2(1 + \xi_0\bar{\xi}_0) + \beta_0)}{1 - 2\xi_0\bar{\xi}_0 - 3\xi_0^2\bar{\xi}_0^2 + (1 - \xi_0\bar{\xi}_0)\beta_0},$$

where we have chosen the + sign in the equations. Further straightforward, if lengthy, calculations establish that

$$\alpha_1 = \frac{1 + \xi_0\bar{\xi}_0 + \beta_0}{1 - 2\xi_0\bar{\xi}_0 - 3\xi_0^2\bar{\xi}_0^2 + \beta_0}, \quad \alpha_2 = \frac{\xi_0(3(1 + \xi_0\bar{\xi}_0) + \beta_0)}{1 - 2\xi_0\bar{\xi}_0 - 3\xi_0^2\bar{\xi}_0^2 + \beta_0},$$

$$\alpha_3 = \frac{\xi_0(1 + \xi_0\bar{\xi}_0 + \beta_0)}{(1 - 2\xi_0\bar{\xi}_0 - 3\xi_0^2\bar{\xi}_0^2 + \beta_0)^2}.$$

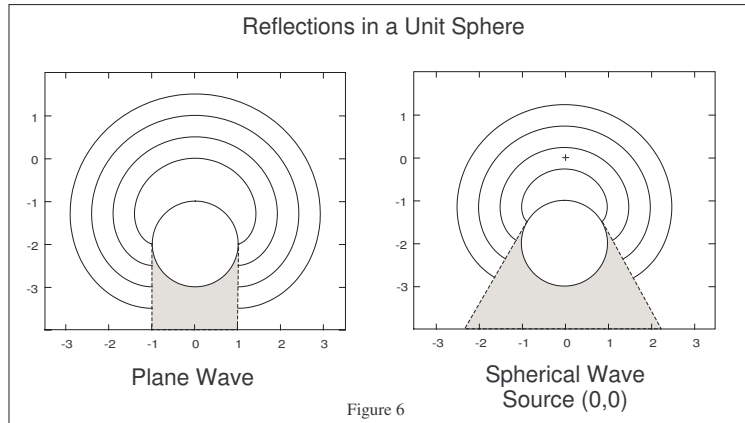
Finally, substitution of these in (6.2) gives

$$\eta_2 = \frac{4\xi_0(1 - 3\xi_0\bar{\xi}_0)(1 + 3\xi_0\bar{\xi}_0 + \sqrt{1 + 10\xi_0\bar{\xi}_0 + 9\xi_0^2\bar{\xi}_0^2})}{1 + \xi_0\bar{\xi}_0 - 7\xi_0^2\bar{\xi}_0^2 + 9\xi_0^3\bar{\xi}_0^3 + (1 - 4\xi_0\bar{\xi}_0 + 3\xi_0^2\bar{\xi}_0^2)\sqrt{1 + 10\xi_0\bar{\xi}_0 + 9\xi_0^2\bar{\xi}_0^2}}.$$

To find the potential function r_2 we invert (2.4) with $\nu = \xi_0$, since we are parameterising the outgoing rays by the direction of the normal to the surface at the point of intersection. After some computation we obtain

$$r_2 = \frac{-2(1 - 3\xi_0\bar{\xi}_0)^2\sqrt{1 + 10\xi_0\bar{\xi}_0 + 9\xi_0^2\bar{\xi}_0^2}}{1 + 11\xi_0\bar{\xi}_0 + 19\xi_0^2\bar{\xi}_0^2 + 9\xi_0^3\bar{\xi}_0^3} + C.$$

At this juncture we have found parametric expressions for ξ_2 , η_2 and r_2 , that is, the complete description of the outgoing wave. If we graph the wavefronts the results are similar to the plane wave reflected off the sphere. By way of comparison, Figure 6 compares a cross-section of the two wavefronts in a plane containing the x^3 -axis. The shaded area represents the shadow cast by the surface.



REFERENCES

- [1] V.I. Arnol'd, S.M. Guesin-Zade and A.N. Varchenko *Singularities of differential mappings Vol. 1*, Birkhäuser Boston Inc., Boston, MA 1985, The classification of critical points, caustics and wave fronts, translated from the Russian by Ian Porteous and Mark Reynolds.
- [2] B. Guilfoyle and W. Klingenberg, *On the space of oriented affine lines in \mathbb{R}^3* , Archiv der Math. **82** (2004), 81-84.
- [3] B. Guilfoyle and W. Klingenberg, *Generalised surfaces in \mathbb{R}^3* , Math. Proc. of the R.I.A. **104A(2)** (2004) [math.DG/0406185].
- [4] N.J. Hitchin, *Monopoles and geodesics*, Comm. Math. Phys. **83** (1982), no. 4, 579-602.
- [5] V. Hlavaty, *Differential line geometry*, Noordhoff, Groningen, 1953.
- [6] J. Hoschek, *Liniengeometrie*, Bibliographisches Institut Zuerich, 1971.
- [7] H. Pottner, J. Wallner, *Computational Line Geometry*, Springer-Verlag, Berlin 2001.
- [8] K. Weierstrass, *Untersuchungen über die Flächen, deren mittlere Krümmung überall gleich Null ist*, Monatsber. Akad. Wiss. Berlin (1866), 612-625.
- [9] E. T. Whittaker, *On the partial differential equations of mathematical physics*, Math. Ann. **57** (1903), 333-355.

BRENDAN GUILFOYLE, DEPARTMENT OF MATHEMATICS AND COMPUTING, INSTITUTE OF TECHNOLOGY, TRALEE, CLASH, TRALEE, CO. KERRY, IRELAND.

E-mail address: `brendan.guilfoyle@ittralee.ie`

WILHELM KLINGENBERG, DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF DURHAM, DURHAM DH1 3LE, UNITED KINGDOM.

E-mail address: `wilhelm.klingenberg@durham.ac.uk`