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HOPF HYPERSURFACES IN SPACES OF ORIENTED GEODESICS

NIKOS GEORGIOU AND BRENDAN GUILFOYLE

Abstract. A Hopf hypersurface in a (para-)Kaehler manifold is a real hypersurface for which one of the principal directions of the second fundamental form is the (para-)complex dual of the normal vector.

We consider particular Hopf hypersurfaces in the space of oriented geodesics of a non-flat space form of dimension greater than 2. For spherical and hyperbolic space forms, the oriented geodesic space admits a canonical Kaehler-Einstein and para-Kaehler-Einstein structure, respectively, so that a natural notion of a Hopf hypersurface exists.

The particular hypersurfaces considered are formed by the oriented geodesics that are tangent to a given convex hypersurface in the underlying space form. We prove that a tangent hypersurface is Hopf in the space of oriented geodesics with respect to this canonical (para-)Kaehler structure iff the underlying convex hypersurface is totally umbilic and non-flat.

In the case of 3 dimensional space forms, however, there exists a second canonical complex structure which can also be used to define Hopf hypersurfaces. We prove that in this dimension, the tangent hypersurface of a convex hypersurface in the space form is always Hopf with respect to this second complex structure.

Contents

1. Background and Results

Submanifold theory and in particular the study of real hypersurfaces in a complex manifold, has been of great interest for the last decades (for further study see [6] and [13]). Let (M, g, J) be a Kähler structure, where M is a $2n$-real dimensional manifold, g stands for the pseudo-Riemannian metric and J denotes either a complex or paracomplex structure. If Σ is a non-degenerate real hypersurface of M then there exists a unit normal vector field N along Σ. The *structure vector field* of Σ is the tangential vector field ξ given by $\xi := -JN$. A *Hopf hypersurface* is a

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real hypersurface in a Kähler manifold whose structure vector field is a principal direction.

The principal curvature associated to the structure vector field is called a *Hopf principal curvature*. For Riemannian complex space forms, Madea in [14], Ki and Suh in [12], have proved that the Hopf principal curvature in a Hopf hypersurface must be constant. The same statement for pseudo-Riemannian complex space forms and for paracomplex space forms has been proved recently by Anciaux and Panagiotidou in [3]. Furthermore, depending on the size of the Hopf principal curvature, a local characterization of Hopf hypersurfaces is obtained in complex space forms [3] [5] [15].

The space $\mathbb{L}(\mathbb{S}^{n+1}_\epsilon)$ of oriented geodesics of a real space form $\mathbb{S}^{n+1}_\epsilon$ provides a new class of (para-) complex manifolds for $n\geq 2$. Here $\mathbb{S}^{n+1}_\epsilon$ is the round $(n+1)$-sphere $\mathbb{S}^{n+1}$ when $\epsilon=1$ while, for $\epsilon=-1$ the real space form $\mathbb{S}^{n+1}_\epsilon$ is the hyperbolic $(n+1)$-space $\mathbb{H}^{n+1}$.

In particular, $\mathbb{L}(\mathbb{S}^{n+1}_1)$ admits a canonical Kähler structure $(\mathbb{J},\mathbb{G})$, where $\mathbb{J}$ is a complex structure and $\mathbb{L}(\mathbb{S}^{n+1}_{-1})$ admits a canonical para-Kähler structure which will be also denoted by $(\mathbb{J},\mathbb{G})$.

In both cases, the metric $\mathbb{G}$ is Einstein and together with $\mathbb{J}$ are both invariant under the natural action of the group of isometries of $\mathbb{S}^{n+1}_\epsilon$ (see [1] and [2]). The relation between submanifold theory of $\mathbb{S}^{n+1}_\epsilon$ and $\mathbb{L}(\mathbb{S}^{n+1}_\epsilon)$ has been explored by several authors recently (see [2] [4] [8] [9] and [10]). For example, the Gauss map of hypersurfaces in $\mathbb{S}^{n+1}_\epsilon$ correspond to Lagrangian submanifolds in $\mathbb{L}(\mathbb{S}^{n+1}_\epsilon)$.

The purpose of this paper is to study hypersurfaces in $\mathbb{L}(\mathbb{S}^{n+1}_\epsilon)$ that are formed by the oriented geodesics tangent to a submanifold in $\mathbb{S}^{n+1}_\epsilon$, called *tangent hypersurfaces*. These hypersurfaces were introduced in [11] and further explored in [7].

In particular, we study Hopf tangent hypersurfaces in $(\mathbb{L}(\mathbb{S}^{n+1}_\epsilon),\mathbb{J},\mathbb{G})$ and we prove the following:

Theorem 1. *The tangent hypersurface $\mathcal{H}(\Sigma)$ of an n-dimensional submanifold $\Sigma\subset\mathbb{S}^{n+1}$ (resp. hyperbolic space $\mathbb{H}^{n+1}$) for $n\geq 2$ is a Hopf hypersurface of $(\mathbb{L}(\mathbb{S}^{n+1}),\mathbb{J},\mathbb{G})$ (resp. $(\mathbb{L}(\mathbb{H}^{n+1}),\mathbb{J},\mathbb{G})$) iff it is totally umbilic and non-flat.*

In 3 dimensions, the space $\mathbb{L}(\mathbb{S}^3_\epsilon)$ admits a second canonical complex structure, $\mathbb{J}'$, which is also invariant under the natural action of the group of isometries of $\mathbb{S}^3_\epsilon$. Using $\mathbb{J}'$ it is possible to obtain another invariant metric $\overline{\mathbb{G}}$ on $\mathbb{L}(\mathbb{S}^3_\epsilon)$ (see equation (2.1)). The metric $\overline{\mathbb{G}}$ is of neutral signature and is locally conformally flat [2] [16].

Tangent hypersurfaces in $\mathbb{L}(\mathbb{S}^3_\epsilon)$ have been studied studied using the neutral metric $\overline{\mathbb{G}}$ in [7]. In particular, the tangent hypersurface of an embedded strictly convex 2-sphere is null, i.e., the unit normal vector field has zero length with respect to the neutral metric. Furthermore, the totally null planes form a pair of plane fiels on the tangent hypersurface that are contact.

Regarding the Einstein metric $\mathbb{G}$ we show:

Theorem 2. *Let S be a smooth closed convex surface in $\mathbb{S}^3_\epsilon$. Then the tangent hypersurface $\mathcal{H}(S)$ is a Hopf hypersurface of $(\mathbb{L}(\mathbb{S}^3_\epsilon),\mathbb{J}',\mathbb{G})$.*

In the next secion we establish notation and preliminaries, while Section 3 contains the proof of Theorem 1. The proof of Theorem 2 is in Section 4.

2. Notation and Preliminaries

We adopt the notation of section 3.2 of [7], extended to higher dimensions as in [2].

Let $\mathbb{S}^{n+1}_\epsilon = \{x \in \mathbb{R}^{n+2} \ : \ \langle x, x\rangle_\epsilon = 1\}$ be the $(n+1)$-(pseudo)-sphere in the Euclidean space $\mathbb{R}^{n+2}_\epsilon := (\mathbb{R}^{n+2}, \langle ., .\rangle_\epsilon)$ for $n \geq 2$. Note that $\mathbb{S}^{n+1}_1$ is the round $(n+1)$-sphere $\mathbb{S}^{n+1}$, while $\mathbb{S}^{n+1}_{-1}$ is anti-isometric to the hyperbolic $(n+1)$-space $\mathbb{H}^{n+1}$.

The space of oriented geodesics $\mathbb{L}(\mathbb{S}^{n+1}_\epsilon) \subset \Lambda^2(\mathbb{R}^{n+2})$ of $(\mathbb{S}^{n+1}_\epsilon, g_\epsilon)$ is $2n$-dimensional and $\mathbb{L}(\mathbb{S}^{n+1}_1)$ can be identified with the Grasmannian of oriented planes in $\mathbb{R}^{n+2}_1$, while $\mathbb{L}(\mathbb{S}^{n+1}_{-1})$ can be identified with the Grasmannian of oriented planes in $\mathbb{R}^{n+2}_{-1}$ such that the induced metric is Lorentzian [2].

Recall the complex (resp. paracomplex) structure $\mathbb{J}_\epsilon$ on $\mathbb{L}(\mathbb{S}^{n+1}_\epsilon)$ defined by:

$$\mathbb{J}_\epsilon : T_{x\wedge y}\mathbb{L}(\mathbb{S}^{n+1}_\epsilon) \to: T_{x\wedge y}\mathbb{L}(\mathbb{S}^{n+1}_\epsilon) : x \wedge X + y \wedge Y \mapsto y \wedge X - x \wedge Y,$$

and simply write $\mathbb{J}$ for $\mathbb{J}_\epsilon$. Finally, consider the $SO(n+2)$ (resp. $SO(1, n+1)$)-invariant Einstein metric $\mathbb{G}_\epsilon$, given by

$$\mathbb{G}_\epsilon = \iota^\star \langle\langle ., .\rangle\rangle_\epsilon ,$$

where $\langle\langle ., .\rangle\rangle_\epsilon$ is the flat metric of $\Lambda^2(\mathbb{R}^{n+2})$. Then, $(\mathbb{L}(\mathbb{S}^{n+1}), \mathbb{J}, \mathbb{G})$ (resp. $(\mathbb{L}(\mathbb{H}^{n+1}), \mathbb{J}, \mathbb{G})$) is a (resp. para-) Kähler structure [1] [2] [8].

The four-dimensional manifold $\mathbb{L}(\mathbb{S}^3_\epsilon)$ enjoys other natural complex structure, which is defined as follows: the orthogonal two-plane $(x \wedge y)^\perp$ is Riemannian and admits a canonical orientation (that orientation compatible with the orientations of $x \wedge y$ and $\mathbb{R}^4$). Thus it enjoys a canonical complex structure J'. The following endomorphism

$$\mathbb{J}'(x \wedge X + y \wedge Y) := x \wedge (J'X) + y \wedge (J'Y),$$

defines another complex structure on $\mathbb{L}(\mathbb{S}^3_\epsilon)$ that is compatible with $\mathbb{G}$. Thus, $(\mathbb{L}(\mathbb{S}^3_\epsilon), \mathbb{G}, \mathbb{J}')$ is another Kähler structure (see [1] [2] [4]). Since $\mathbb{J}$ and $\mathbb{J}'$ commute, we may define the following metric on $\mathbb{L}(\mathbb{S}^3_\epsilon)$:

$$\overline{\mathbb{G}}(\cdot, \cdot) = -\epsilon\mathbb{G}(\cdot, \mathbb{J} \circ \mathbb{J}'\cdot), \tag{2.1}$$

which is of neutral signature and locally conformally flat.

Definition 1. A *tangent hypersurface* $\mathcal{H}(\Sigma)$ over a hypersurface Σ in $\mathbb{S}^{n+1}_\epsilon$ is the hypersurface of $\mathbb{L}(\mathbb{S}^{n+1}_\epsilon)$ formed by the oriented geodesics in $\mathbb{S}^{n+1}_\epsilon$ tangent to Σ at some point.

This was introduced for $n = 2$ in the flat case in [11] and the curved case in [7]. In this dimension $\mathcal{H}(S)$ is $\overline{\mathbb{G}}$-null, i.e., the unit normal vector field is of zero length with respect to the metric $\overline{\mathbb{G}}$. Furthermore, $\mathcal{H}(S)$ is locally a circle bundle over S, with projection $\pi : \mathcal{H}(S) \to S$ and fibre generated by rotation about the normal to S. For further details and properties in this dimension, see [7].

3. Hopf Tangent Hypersurfaces

In this section we consider the conditions under which a tangent hypersurface is Hopf with respect to the canonical (para-)Kaehler structure $(\mathbb{J},\mathbb{G})$.

We start with the following Lemma:

Lemma 1. *Let $(e_1,\ldots,e_n)$ be an orthonormal basis of $\mathbb{R}^n$. Then, for every $v\in\mathbb{S}^{n-1}\subset\mathbb{R}^n$ there exist $\theta_1\in[0,2\pi)$ and $\theta_2,\ldots,\theta_{n-1}\in[-\pi/2,\pi/2]$, such that*

$$v=\cos\theta_1\ldots\cos\theta_{n-1}\,e_1+\sin\theta_1\cos\theta_2\ldots\cos\theta_{n-1}\,e_2+\sin\theta_2\cos\theta_3\ldots\cos\theta_{n-1}\,e_3+$$
$$\ldots+\sin\theta_{n-2}\cos\theta_{n-1}\,e_{n-1}+\sin\theta_{n-1}\,e_n.$$

Proof. Since $\langle e_i,e_j\rangle=\delta_{ij}$, every vector v in $\mathbb{R}^n$ satisfies

$$\langle v,v\rangle=\langle v,e_1\rangle\langle v,e_1\rangle+\ldots+\langle v,e_n\rangle\langle v,e_n\rangle,$$

and the fact that $v\in\mathbb{S}^{n-1}$ yields,

$$\langle v,e_1\rangle^2+\ldots+\langle v,e_n\rangle^2=1. \tag{3.1}$$

Then,

$$|\langle v,e_n\rangle|\leq 1,$$

Thus, there exists $\theta_{n-1}\in[-\pi/2,\pi/2]$ such that

$$\langle v,e_n\rangle=\sin\theta_{n-1}. \tag{3.2}$$

Using (3.2), we get,

$$\langle v,e_1\rangle^2+\ldots+\langle v,e_{n-1}\rangle^2=\cos^2\theta_{n-1}. \tag{3.3}$$

If $|\theta_{n-1}|=\pi/2$, we have

$$\langle v,e_1\rangle=\ldots=\langle v,e_{n-1}\rangle=0,$$

and choosing $\theta_1=\ldots=\theta_{n-2}=0$, we obtain $v=e_n$. Similar argument shows that if $|\theta_k|=\pi/2$ for some k, then $\theta_i=0$ for all $i<k$.

Suppose that $|\theta_k|\neq\pi/2$ for all k. Following (3.3) we have

$$\left(\frac{\langle v,e_1\rangle}{\cos\theta_{n-1}}\right)^2+\ldots+\left(\frac{\langle v,e_{n-1}\rangle}{\cos\theta_{n-1}}\right)^2=1.$$

We then have

$$\left|\frac{\langle v,e_{n-1}\rangle}{\cos\theta_{n-1}}\right|\leq 1$$

and so there exists $\theta_{n-2}\in[-\pi/2,\pi/2]$ such that

$$\frac{\langle v,e_{n-1}\rangle}{\cos\theta_{n-1}}=\sin\theta_{n-2}.$$

It follows,

$$\langle v,e_{n-1}\rangle=\sin\theta_{n-2}\cos\theta_{n-1}. \tag{3.4}$$

From (3.4), we obtain

$$\left(\frac{\langle v,e_1\rangle}{\cos\theta_{n-1}}\right)^2+\ldots+\left(\frac{\langle v,e_{n-2}\rangle}{\cos\theta_{n-1}}\right)^2=\cos^2\theta_{n-2},$$

which yields,

$$\left(\frac{\langle v,e_1\rangle}{\cos\theta_{n-2}\cos\theta_{n-1}}\right)^2+\ldots+\left(\frac{\langle v,e_{n-2}\rangle}{\cos\theta_{n-2}\cos\theta_{n-1}}\right)^2=1,$$

and hence there exists $\theta_{n-3} \in [-\pi/2, \pi/2]$ such that

$$\frac{\langle v, e_{n-1}\rangle}{\cos\theta_{n-2}\cos\theta_{n-1}} = \sin\theta_{n-3}.$$

Equivalently,

$$\langle v, e_{n-2}\rangle = \sin\theta_{n-3}\cos\theta_{n-2}\cos\theta_{n-1}.$$

Applying the same process we obtain angles $\theta_2, \dots, \theta_{n-1} \in [-\pi/2, \pi/2]$, satisfying

$$\langle v, e_k\rangle = \sin\theta_{k-1}\cos\theta_k \dots \cos\theta_{n-1}, \quad k = 3, \dots, n.$$

We then have,

$$\left(\frac{\langle v, e_1\rangle}{\cos\theta_2 \dots \cos\theta_{n-1}}\right)^2 + \left(\frac{\langle v, e_2\rangle}{\cos\theta_2 \dots \cos\theta_{n-1}}\right)^2 = 1.$$

Thus, there exists $\theta \in [0, 2\pi)$, such that

$$\frac{\langle v, e_1\rangle}{\cos\theta_2 \dots \cos\theta_{n-1}} = \cos\theta_1 \qquad \frac{\langle v, e_2\rangle}{\cos\theta_2 \dots \cos\theta_{n-1}} = \sin\theta_1,$$

and the lemma follows. □

Definition 2. Let (M, g) be a smooth manifold and Σ be a hypersurface in M. A point $x \in \Sigma$ is said to be *umbilic* if the second fundamental form h is proportional to the first fundamental form, i.e. there exists a constant λ such that

$$h(X, Y) = \lambda g(X, Y).$$

A hypersurface is said to be *totally umbilic* if all its points are umbilic. In particular, for every point in a totally umbilic hypersurface all principal curvatures are equal.

Proof of Theorem 1: Any vector field X in Σ is identified with $d\phi(X)$ and let $e_1, \dots, e_n$ be the principal directions of ϕ with corresponding principal curvatures $\lambda_1, \dots, \lambda_n$. Using Lemma 1, the tangent hypersurface $\mathcal{H}(\Sigma)$ can be locally parametrized by

$$\bar{\phi} : \Sigma \times \mathbb{S}^{n-1} \to \mathbb{L}(\mathbb{S}^{n+1}) : (x, \theta_1, \dots, \theta_{n-1}) \mapsto \phi(x) \wedge v(x, \theta_1, \dots, \theta_{n-1}),$$

where,

$$v = \cos\theta_1 \dots \cos\theta_{n-1} e_1(x) + \sin\theta_1\cos\theta_2 \dots \cos\theta_{n-1} e_2(x) + \sin\theta_2\cos\theta_3 \dots \cos\theta_{n-1} e_3(x) + \dots + \sin\theta_{n-2}\cos\theta_{n-1}\, e_{n-1}(x) + \sin\theta_{n-1}\, e_n(x)$$

For $k = 1, \dots, n-1$ define,

$$v_k = \frac{\partial_{\theta_k} v}{|\partial_{\theta_k} v|}.$$

Then,

$$\begin{aligned} v_k = &-\cos\theta_1 \dots \cos\theta_{k-1}\sin\theta_k\, e_1 - \sin\theta_1\cos\theta_2 \dots \cos\theta_{k-1}\sin\theta_k\, e_2 - \\ &- \sin\theta_2\cos\theta_3 \dots \cos\theta_{k-1}\sin\theta_k\, e_3 - \sin\theta_3\cos\theta_4 \dots \cos\theta_{k-1}\sin\theta_k\, e_4 - \\ &- \sin\theta_{k-2}\cos\theta_{k-1}\sin\theta_k\, e_{k-1} - \sin\theta_{k-1}\sin\theta_k\, e_k + \cos\theta_k\, e_{k+1}. \end{aligned}$$

Setting $v_n := v$, one can show that $\langle v_i, v_j\rangle = \delta_{ij}$.

The tangent space $T_{\phi\wedge v}\mathbb{L}(\mathbb{S}^{n+1}_\epsilon)$ on the oriented plane $\phi \wedge v$ in $\mathbb{R}^{n+2}$ is identified with the space of the vector fields that are of the form

$$\phi \wedge X + v \wedge Y,$$

where $X,Y\in(\phi\wedge v)^{\perp}=\text{span}\{N,v_1,\dots,v_{n-1}\}$. Using the (para-) complex structure J defined by $J\phi=v$ and $Jv=-\epsilon\phi$, the (para-) complex structure $\mathbb{J}$ on $\mathbb{L}(\mathbb{S}^{n+1}_\epsilon)$ is defined as follows,

$$\mathbb{J}(\phi\wedge X+v\wedge Y)=(J\phi)\wedge X+(Jv)\wedge Y=-\epsilon\phi\wedge Y+v\wedge X.$$

Consider the matrix $(g_{ij})\in SO(n)$, given by $v_k=\sum_{l=1}^n g_{kl}e_l$ and denote the inverse matrix by (g^{ij}). It then follows,

$$\begin{aligned} d\bar\phi(e_k) &= d(\phi\wedge v)(e_k)\\ &= e_k\wedge v+\phi\wedge\overline{\nabla}_{e_k}v\\ &= \sum_{l=1}^{n-1}g^{kl}v_l\wedge v+\phi\wedge\overline{\nabla}_{e_k}v. \end{aligned}$$

A brief computations gives,

$$\overline{\nabla}_{e_k}v=\sum_{l=1}^{n-1}\sum_{s=1}^{n}g^{ks}\left\langle\overline{\nabla}_{v_s}v,v_l\right\rangle v_l+\left\langle\overline{\nabla}_{e_k}v,\phi\right\rangle\phi+\lambda_k g_{nk}N.$$

Therefore, the tangent bundle $T\mathcal{H}(\Sigma)$ is generated by the vector fields,

$$d\bar\phi(e_k)=\sum_{l=1}^{n-1}g^{kl}v_l\wedge v+\sum_{l=1}^{n-1}\sum_{s=1}^{n}g^{ks}\left\langle\overline{\nabla}_{v_s}v,v_l\right\rangle\phi\wedge v_l+\lambda_k g^{kn}\phi\wedge N. \tag{3.5}$$

The unit normal vector field $\bar N$ of $\mathcal{H}(\Sigma)$ in $\mathbb{L}(\mathbb{S}^{n+1})$ is given by,

$$\bar N=v\wedge N.$$

The structure vector field $\xi=-\mathbb{J}\bar N$ is,

$$\xi=\phi\wedge N.$$

Let $D,\overline{D}$ be the Levi-Civita connection of $\langle\langle .,.\rangle\rangle$ and $\mathbb{G}$, respectively. Then,

$$\begin{aligned} \overline{D}_{d\bar\phi(e_k)}\bar N &= \overline{D}_{d\bar\phi(e_k)}(\phi\wedge N)\\ &= \sum_{l=1}^{n-1}\left\langle\overline{\nabla}_{e_k}v,v_l\right\rangle v_l\wedge N+\left\langle\overline{\nabla}_{e_k}v,\phi\right\rangle\phi\wedge N+\lambda_k\sum_{l=1}^{n-1}g^{kl}v_l\wedge v. \end{aligned}$$

Thus,

$$D_{d\bar\phi(e_k)}\bar N=-g_{nk}\phi\wedge N+\lambda_k\sum_{l=1}^{n-1}g^{kl}v_l\wedge v. \tag{3.6}$$

Similarly,

$$\overline{D}_{d\bar\phi(\partial/\partial\theta_k)}\bar N=\overline{D}_{d\bar\phi(\partial/\partial\theta_k)}(v\wedge N)=(\partial_{\theta_k}v)\wedge N,$$

which gives,

$$D_{d\bar\phi(\partial/\partial\theta_k)}\bar N=0. \tag{3.7}$$

If A stands for the shape operator of $\mathcal{H}(\Sigma)$ in $\mathbb{L}(\mathbb{S}^{n+1})$, the relations (3.6) and (3.7) give,

$$A(d\bar{\phi}(e_k)) = -g_{nk}\phi\wedge N + \lambda_k\sum_{l=1}^{n-1} g^{kl}v_l\wedge v \tag{3.8}$$

$$A(d\bar{\phi}(\partial/\partial\theta_k)) = 0. \tag{3.9}$$

Suppose that all principal curvatures $\lambda_1,\dots,\lambda_n$ are all equal to λ, where $\lambda(x)\neq 0$ for all $x\in\Sigma$. Using (3.5) and the fact that we have,

$$\begin{aligned}\sum_{k=1}^{n} g_{nk}d\bar{\phi}(e_k) &= \sum_{k=1}^{n}\sum_{l=1}^{n-1} g_{nk}g^{kl}v_l\wedge v + \sum_{k=1}^{n}\sum_{l=1}^{n-1}\sum_{s=1}^{n} g_{nk}g^{ks}\left\langle\overline{\nabla}_{v_s}v, v_l\right\rangle\phi\wedge v_l\\ &\quad +\sum_{k=1}^{n}\lambda_k g_{nk}g^{kn}\phi\wedge N\\ &= \sum_{l=1}^{n-1}\left\langle\overline{\nabla}_v v, v_l\right\rangle\phi\wedge v_l + \sum_{k=1}^{n}\lambda g_{nk}g^{kn}\phi\wedge N\\ &= \sum_{l=1}^{n-1}\left\langle\overline{\nabla}_v v, v_l\right\rangle\phi\wedge v_l + \lambda\xi.\end{aligned}$$

The expression,

$$\phi\wedge v_k = \frac{d\phi(\partial/\partial\theta_k)}{|\partial_{\theta_k}v|},$$

gives,

$$\sum_{k=1}^{n} g_{nk}d\bar{\phi}(e_k) = \sum_{k=1}^{n-1}\frac{\left\langle\overline{\nabla}_v v, v_k\right\rangle}{|\partial_{\theta_k}v|}d\bar{\phi}(\partial/\partial\theta_k) + \lambda\xi.$$

Hence,

$$\xi = \lambda^{-1}\sum_{k=1}^{n}\left(g_{nk}d\bar{\phi}(e_k) - \frac{\left\langle\overline{\nabla}_v v, v_k\right\rangle}{|\partial_{\theta_k}v|}d\bar{\phi}(\partial/\partial\theta_k)\right).$$

Using (3.8) and (3.9), we finally get

$$\begin{aligned}A\xi &= \lambda^{-1}\sum_{k=1}^{n}\left(g_{nk}A(d\bar{\phi}(e_k)) - \frac{\left\langle\overline{\nabla}_v v, v_k\right\rangle}{|\partial_{\theta_k}v|}A(d\bar{\phi}(\partial/\partial\theta_k))\right)\\ &= \lambda^{-1}\sum_{k=1}^{n}\left(-g_{nk}g_{nk}\phi\wedge N + \lambda_k\sum_{l=1}^{n-1} g_{nk}g^{kl}v_l\wedge v\right)\\ &= -\lambda^{-1}\left(\sum_{k=1}^{n} g_{nk}^2\right)\xi\\ &= -\lambda^{-1}\xi,\end{aligned}$$

which shows that $\mathcal{H}(\Sigma)$ is a Hopf hypersurface.

Suppose that $\mathcal{H}(\Sigma)$ is Hopf with respect to $(\mathbb{G},\mathbb{J})$. Assuming that ϕ is not totally umbilic, consider the case where the principal curvatures λ_k are all equal to λ except

$\lambda_{k_0}\neq\lambda$. A brief computation gives,

$$\left(\sum_{k=1}^{n}\lambda_k g_{nk}g^{kn}\right)A\xi=\xi+(\lambda_s-\lambda)g_{ns}\left(\sum_{l=1}^{n-1}g^{sl}v_l\right)\wedge v,$$

and shows that $\mathcal{H}(\Sigma)$ is not Hopf. Similar arguments can be used for the cases where two or more principal curvatures differ and the Theorem follows. □

4. The Special Case of Dimension 3

As mentioned in the introduction, 3 dimensional non-flat space forms are unusual in that their exists a second complex structure $\mathbb{J}'$ on the space of oriented geodesics. In this section we consider the conditions under which a tangent hypersurface is Hopf with respect to the Kaehler structure $(\mathbb{J}',\mathbb{G})$.

Using the terminology introduced in Section 2 for dimension 3, we now prove Theorem 2:

Proof of Theorem 2: Let $\phi:S\to\mathbb{S}^3_\epsilon$ be an embedding of a closed convex surface S in $\mathbb{S}^3_\epsilon$ and let (e_1,e_2) be the principal directions with corresponding pricipal curvatures λ_1,λ_2. Let N be the unit normal vector field along the surface $\phi(S)$ such that (ϕ,e_1,e_2,N) is an oriented orthonormal frame of $\mathbb{R}^4$. For $\theta\in\mathbb{S}^1$, define the following tangential vector fields

$$v(x,\theta)=\cos\theta\, e_1(x)+\sin\theta\, e_2(x)\quad\text{and}\quad v^{\perp}(x,\theta)=-\sin\theta\, e_1(x)+\cos\theta\, e_2(x)$$

The tangent hypersurface $\mathcal{H}(S)$ over S is locally parametrised by

$$\begin{array}{rcl}\bar\phi:S\times\mathbb{S}^1 & \to & \mathbb{L}(\mathbb{S}^3_\epsilon)\\ (x,\theta) & \mapsto & \phi(x)\wedge v(x,\theta)\end{array}$$

Let ξ' be the structure vector field of $\mathcal{H}(S)$ with respect to $(\mathbb{J}',\mathbb{G})$, that is, $\xi'=-\mathbb{J}'\bar N$.

Considering the principal directions (e_1,e_2) with pricipal curvatures λ_1,λ_2, the derivative of $\bar\phi$ is given by:

$$\begin{array}{rcl}d\bar\phi(e_1) & = & v_1\,\phi\wedge v^{\perp}+\lambda_1\cos\theta\,\phi\wedge N+\sin\theta\, v\wedge v^{\perp}\\ d\bar\phi(e_2) & = & v_2\,\phi\wedge v^{\perp}+\lambda_2\sin\theta\,\phi\wedge N-\cos\theta\, v\wedge v^{\perp}\\ d\bar\phi(\partial/\partial\theta) & = & \phi\wedge v^{\perp},\end{array}$$

for some smooth functions v_1 and v_2. Clearly, $\mathcal{H}$ is non-degenerate, with respect to $\mathbb{G}$, and the orthonormal normal vector field $\bar N$ is given by

$$\bar N=v\wedge N.$$

Let $\bar D,D$ be the Levi-Civita connections of $\langle\langle .,.\rangle\rangle_\epsilon$ and $\mathbb{G}$, respectively. Denote by A and h the shape operator and the second fundamental form of $\bar\phi$ and let $\bar h$ be the second fundamental form of the inclusion map $\iota:\mathbb{L}(\mathbb{S}^3_\epsilon)\hookrightarrow\Lambda^2(\mathbb{R}^4)$. Note that for any vector fields X,Y of $\mathcal{H}(S)$, we have:

$$\mathbb{G}(h(X,Y),\bar N)=\mathbb{G}(AX,Y).$$

It follows,

$$-\bar D_{d\bar\phi(e_1)}\bar N\quad=\quad -v_1\, v^{\perp}\wedge N+\cos\theta\,\phi\wedge N-\lambda_1\sin\theta\, v\wedge v^{\perp}$$

Now,

$$A(d\bar{\phi}(e_1)) \quad = \quad -\bar{D}_{d\bar{\phi}(e_1)}\bar{N} + \bar{h}(d\bar{\phi}(e_1), \bar{N}),$$

which yields,

$$A(d\bar{\phi}(e_1)) = \cos\theta\,\phi\wedge N - \lambda_1 \sin\theta\, v\wedge v^{\perp}. \tag{4.1}$$

Similarly we get,

$$A(d\bar{\phi}(e_2)) = \sin\theta\,\phi\wedge N + \lambda_2\cos\theta\, v\wedge v^{\perp} \qquad A(d\bar{\phi}(\partial/\partial\theta)) = 0. \tag{4.2}$$

Using (4.1), we have

$$\mathbb{G}(h(d\bar{\phi}(e_1), d\bar{\phi}(e_1)), \bar{N}) \quad = \quad \lambda_1\cos 2\theta.$$

Analogously we have,

$$\mathbb{G}(h(d\bar{\phi}(e_1), d\bar{\phi}(e_2)), \bar{N}) = \mathbb{G}(h(d\bar{\phi}(e_2), d\bar{\phi}(e_1)), \bar{N}) = H\sin 2\theta,$$

$$\mathbb{G}(h(d\bar{\phi}(e_2), d\bar{\phi}(e_2)), \bar{N}) = -\lambda_2\cos 2\theta.$$

and

$$\mathbb{G}(h(d\bar{\phi}(e_1), d\bar{\phi}(\partial/\partial\theta)), \bar{N}) = \mathbb{G}(h(d\bar{\phi}(e_2), d\bar{\phi}(\partial/\partial\theta)), \bar{N}) = 0,$$

$$\mathbb{G}(h(d\bar{\phi}(\partial/\partial\theta, d\bar{\phi}(\partial/\partial\theta)), \bar{N}) = 0.$$

In terms of $(e_0 := d\bar{\phi}(\partial/\partial\theta), d\bar{\phi}(e_1), d\bar{\phi}(e_2))$, the second fundamental form h can be expressed by the following symmetric matrix

$$h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_1\cos 2\theta & H\sin 2\theta \\ 0 & H\sin 2\theta & -\lambda_2\cos 2\theta \end{pmatrix}$$

The principal curvatures are the eigenvalues of h, which are $0, \lambda_+$ and λ_-, where

$$\lambda_+ = \lambda_1\cos^2\theta + \lambda_2\sin^2\theta \qquad \lambda_- = -\lambda_1\sin^2\theta - \lambda_2\cos^2\theta,$$

with corresponding principal directions e_0, v_+ and v_-. Then,

$$v_+ \quad = \quad \cos\theta\, d\bar{\phi}(e_1) + \sin\theta\, d\bar{\phi}(e_2),$$

and thus,

$$v_+ = \left\langle \nabla_v v, v^{\perp}\right\rangle\, \phi\wedge v^{\perp} + (\lambda_1\cos^2\theta + \lambda_2\sin^2\theta)\,\phi\wedge N.$$

The fact that S is closed and convex implies that

$$\lambda_+\lambda_- < 0,$$

and $\{v_+, e_0\}$ are linearly independent. Thus, the principal directions e_0 and v_+ span the α-plane Π_+ [7], that is,

$$\Pi_+ = \mathrm{span}\{e_0, v_+\}.$$

It can be easily proved that

$$\mathbb{J}e_0 = \mathbb{J}'\bar{N} = -\xi'. \tag{4.3}$$

Since $\mathbb{J}\Pi_+ = \Pi_+$, it then follows that $\xi' \in \Pi_+$ and thus ξ' is a principal direction. Hence, $\mathcal{H}(S)$ is a Hopf hypersurface of $(\mathbb{L}(\mathbb{S}^3_\epsilon), \mathbb{J}', \mathbb{G})$. $\square$

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