2007-01-27

Reflection in a Translation Invariant Surface

Brendan Guilfoyle

Wilhelm Klingenberg

REFLECTION IN A TRANSLATION INVARIANT SURFACE

BRENDAN GUILFOYLE AND WILHELM KLINGENBERG

Abstract. We prove that the focal set generated by the reflection of a point source off a translation invariant surface consists of two sets: a curve and a surface. The focal curve lies in the plane orthogonal to the symmetry direction containing the source, while the focal surface is translation invariant.

This is done by constructing explicitly the focal set of the reflected line congruence (2-parameter family of oriented lines in $\mathbb{R}^3$) with the aid of the natural complex structure on the space of all oriented affine lines.

The purpose of this paper is to prove the following Theorem:

Main Theorem:

The focal set generated by the reflection of a point source off a translation invariant surface consists of two sets: a curve and a surface. The focal curve lies in the plane orthogonal to the symmetry direction containing the source, while the focal surface is translation invariant.

In contrast to the focal surface, the reflected wavefront is not translation invariant, in general.

There have been many investigations of generic focal sets of line congruences [1] [2] [5]. Rather than work in the generic setting, we compute the focal set explicitly in this special case. This we do by applying recent work on immersed surfaces in the space $\mathbb{T}$ of oriented affine lines in $\mathbb{R}^3$ [3] [4]. The next section contains a summary of the background material on the complex geometry of $\mathbb{T}$ and the focal sets of arbitrary line congruences. It also details the reflection of a line congruence in an oriented surface in $\mathbb{R}^3$.

In Section 2 we solve the problem of reflection of a point source off an arbitrary translation invariant surface (Proposition 2). We then compute the focal set and thus prove the Main Theorem.

1. The Parametric Approach to Geometric Optics

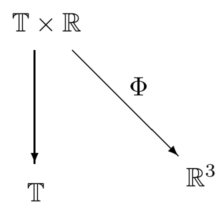
Let $\mathbb{T}$ be the set of oriented affine lines in Euclidean $\mathbb{R}^3$, which, by parallel translation, can be identified with the tangent bundle to the 2-sphere, $\mathrm{T}\mathbb{P}^1$. We summarise now the main features of this identification (further details can be found in [3] [4]).

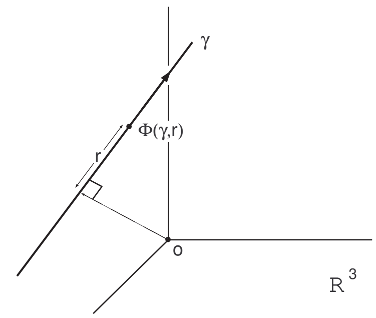
Date: September 26, 2005.

1991 *Mathematics Subject Classification.* 53A25 78A05 53C80.

Key words and phrases. line congruence, focal set, reflection, caustic. PACS: 42.15.-i.

The canonical projection $\pi : \mathbb{T} \to \mathbb{P}^1$ assigns an oriented line, or *ray*, to its direction, and we also have the double fibration:





In the diagram, the lefthand map is projection onto the first factor. The mapping on the right, denoted by Φ, takes $(\gamma, r) \in \mathbb{T} \times \mathbb{R}$ to the point on the oriented line γ in $\mathbb{R}^3$ that lies an affine parameter distance r from the point on γ closest to the origin (as shown).

Let ξ be the local coordinates on $\mathbb{P}^1$ obtained by stereographic projection from the south pole. This can be extended to coordinates (ξ, η) on $\mathbb{T}$ minus the fibre over the south pole. The map $(\xi, \eta, r) \mapsto \Phi(\xi, \eta, r) = (z(\xi, \eta, r), t(\xi, \eta, r))$ has the following cooordinate expression [3]:

$$z = \frac{2(\eta - \bar{\eta}\xi^2) + 2\xi(1 + \xi\bar{\xi})r}{(1 + \xi\bar{\xi})^2} \qquad\qquad t = \frac{-2(\eta\bar{\xi} + \bar{\eta}\xi) + (1 - \xi^2\bar{\xi}^2)r}{(1 + \xi\bar{\xi})^2}, \tag{1.1}$$

where $z = x^1 + ix^2$, $t = x^3$ and $(x^1,\, x^2,\, x^3)$ are Euclidean coordinates in $\mathbb{R}^3$.

Definition 1. A *line congruence* is an immersed surface $f : \Sigma \to \mathbb{T}$, i.e. a 2-parameter family of oriented lines in $\mathbb{R}^3$. A *smoothly parameterised* line congruence is a smoothly immersed surface $f : \Sigma \to \mathbb{T}$ together with an open cover $\{U_\alpha\}$ of Σ and diffeomorphisms $\mathbb{C} \to U_\alpha : \mu_\alpha \mapsto \gamma$. For short we denote a parameterisation simply by μ, and assume that all maps are at least C^1-smooth.

The first order properties of such a family can described by two complex functions, the *optical scalars*: $\rho, \sigma : \Sigma \times \mathbb{R} \to \mathbb{C}$, which are defined relative to an orthonormal frame in $\mathbb{R}^3$ adapted to the congruence. The real part θ and the imaginary part λ of ρ are the *divergence* and *twist* of the congruence, while σ is the *shear* [6].

For a parameterised line congruence we compose with the coordinates above to get $\mu \mapsto (\xi(\mu, \bar{\mu}), \eta(\mu, \bar{\mu}))$. The optical scalars then, with a natural choice of orthonormal frame, have the following expressions [3]:

$$\rho = \theta + \lambda i = \frac{\partial^+\eta\,\bar{\partial}\,\bar{\xi} - \partial^-\eta\,\partial\bar{\xi}}{\partial^-\eta\,\overline{\partial^-\eta} - \partial^+\eta\,\overline{\partial^+\eta}} \qquad\qquad \sigma = \frac{\overline{\partial^+\eta}\,\partial\bar{\xi} - \overline{\partial^-\eta}\,\bar{\partial}\,\bar{\xi}}{\partial^-\eta\,\overline{\partial^-\eta} - \partial^+\eta\,\overline{\partial^+\eta}}, \tag{1.2}$$

where

$$\partial^+\eta \equiv \partial\eta + r\partial\xi - \frac{2\eta\bar{\xi}\partial\xi}{1 + \xi\bar{\xi}} \qquad\qquad \partial^-\eta \equiv \bar{\partial}\eta + r\bar{\partial}\xi - \frac{2\eta\bar{\xi}\,\bar{\partial}\xi}{1 + \xi\bar{\xi}},$$

and ∂ and $\bar{\partial}$ are differentiation with respect to μ and $\bar{\mu}$, respectively.

Definition 2. The *curvature* of a line congruence is defined to be $\kappa = \rho\bar{\rho} - \sigma\bar{\sigma}$. A line congruence is *flat* if $\kappa = 0$. A line congruence $\Sigma \subset \mathbb{T}$ is flat iff the rank of the projection $\pi : \mathbb{T} \to \mathbb{P}^1$ restricted to Σ is non-maximal.

Definition 3. A point p on a line γ in a line congruence is a *focal point* if ρ and σ blow-up at p. The set of focal points of a line congruence Σ generically form surfaces in $\mathbb{R}^3$, which will be referred to as the *focal surfaces* of Σ.

Theorem 1. *The focal set of a parametric line congruence* Σ *is*

$$\{\Phi(\gamma, r)|\ \gamma \in \Sigma \ \ and\ \ 1 - 2\theta_0 r + (\rho_0\overline{\rho}_0 - \sigma_0\overline{\sigma}_0)r^2 = 0\},$$

where the coefficients of the quadratic equation are given by (1.2) at $r = 0$.

Proof. In terms of the affine parameter r along a given line, the Sachs equations, which σ and ρ must satisfy, are [6]:

$$\frac{\partial\rho}{\partial r} = \rho^2 + \sigma\overline{\sigma} \qquad \frac{\partial\sigma}{\partial r} = (\rho + \overline{\rho})\sigma.$$

These are equivalent to the vanishing of certain components of the Ricci tensor of the Euclidean metric. They have solution:

$$\rho = \frac{\rho_0 - (\rho_0\overline{\rho}_0 - \sigma_0\overline{\sigma}_0)r}{1 - 2\theta_0 r + (\rho_0\overline{\rho}_0 - \sigma_0\overline{\sigma}_0)r^2} \qquad \sigma = \frac{\sigma_0}{1 - 2\theta_0 r + (\rho_0\overline{\rho}_0 - \sigma_0\overline{\sigma}_0)r^2},$$

where σ_0, θ_0 and ρ_0 are the values of the optical scalars at $r = 0$. The theorem follows. □

This has the following corollary:

Corollary 1. *Let* Σ *be a line congruence,* $\rho = \theta + \lambda i$*,* σ *the associated optical scalars and* ρ_0*,* θ_0*,* λ_0*,* σ_0 *their values at* $r = 0$*.*

If Σ *is flat with non-zero divergence, then there exists a unique focal surface* S *given by* $r = (2\theta_0)^{-1}$*. If it is flat with zero divergence, then the focal set is empty.*

If Σ *is non-flat, then there exists a unique focal point on each line iff* $|\sigma_0|^2 = \lambda_0^2$*, there exist two focal points on each line iff* $|\sigma_0|^2 < \lambda_0^2$ *and there are no focal points on each line iff* $|\sigma_0|^2 > \lambda_0^2$*. The focal set is given by*

$$r = \frac{\theta_0 \pm (|\sigma_0|^2 - \lambda_0^2)^{\frac{1}{2}}}{\rho_0\bar{\rho}_0 - \sigma_0\bar{\sigma}_0}.$$

Proof. The focal set of a parameterised line congruence are given by $r = r(\mu, \bar{\mu})$ satisfying the quadratic equation in Theorem 1. If $\kappa = 0$, then there is none or one solution depending on whether $\theta_0 = 0$ or not.

If $\kappa \neq 0$ then there are two, one or no solutions iff $|\sigma_0|^2 - \lambda_0^2$ is greater than, equal to or less than zero (respectively).

The solution of the quadratic equation in each case is as stated. □

Given a line congruence $f : \Sigma \to \mathbb{T}$, a map $s : \Sigma \to \mathbb{R}$ determines a map $\Sigma \to \mathbb{R}^3$ by $\gamma \mapsto \Phi(f(\gamma), s(\gamma))$ for $\gamma \in \Sigma$. With a local parameterisation μ of Σ, we get a map $\mathbb{C} \to \mathbb{R}^3$ which comes from substituting $r = s(\mu, \bar{\mu})$ in equations (1.1).

Of particular interest are the surfaces in $\mathbb{R}^3$ orthogonal to the line congruence - when the line congruence is *normal*. These exist iff the twist of the congruence vanishes, and the surfaces are obtained from the solutions of the following equation [3]:

$$\bar{\partial} r = \frac{2\eta\bar{\partial}\bar{\xi} + 2\bar{\eta}\bar{\partial}\xi}{(1+\xi\bar{\xi})^2}. \tag{1.3}$$

We turn now to the reflection of an oriented line in a surface in $\mathbb{R}^3$. This is equivalent to the action of a certain group on the space of oriented lines, as described by [4]:

Theorem 2. *Consider a parametric line congruence $\xi = \xi_1(\mu_1, \bar{\mu}_1)$, $\eta = \eta_1(\mu_1, \bar{\mu}_1)$ reflected off an oriented surface with parameterised normal line congruence $\xi = \xi_0(\mu_0, \bar{\mu}_0)$, $\eta = \eta_0(\mu_0, \bar{\mu}_0)$ and $r = r_0(\mu_0, \bar{\mu}_0)$ satisfying (1.3) with $\xi = \xi_0$ and $\eta = \eta_0$. Then the reflected line congruence is*

$$\xi = \frac{2\xi_0\bar{\xi}_1 + 1 - \xi_0\bar{\xi}_0}{(1-\xi_0\bar{\xi}_0)\bar{\xi}_1 - 2\bar{\xi}_0}, \tag{1.4}$$

$$\eta = \tfrac{(\bar{\xi}_0-\bar{\xi}_1)^2}{((1-\xi_0\bar{\xi}_0)\bar{\xi}_1-2\bar{\xi}_0)^2}\eta_0 - \tfrac{(1+\xi_0\bar{\xi}_1)^2}{((1-\xi_0\bar{\xi}_0)\bar{\xi}_1-2\bar{\xi}_0)^2}\bar{\eta}_0 + \tfrac{(\bar{\xi}_0-\bar{\xi}_1)(1+\xi_0\bar{\xi}_1)(1+\xi_0\bar{\xi}_0)}{((1-\xi_0\bar{\xi}_0)\bar{\xi}_1-2\bar{\xi}_0)^2}r_0, \tag{1.5}$$

where the incoming rays are only reflected if they satisfy the intersection equation

$$\eta_1 = \tfrac{(1+\bar{\xi}_0\xi_1)^2}{(1+\xi_0\bar{\xi}_0)^2}\eta_0 - \tfrac{(\xi_0-\xi_1)^2}{(1+\xi_0\bar{\xi}_0)^2}\bar{\eta}_0 + \tfrac{(\xi_0-\xi_1)(1+\bar{\xi}_0\xi_1)}{1+\xi_0\bar{\xi}_0}r_0. \tag{1.6}$$

By virtue of the intersection equation, an alternative way of writing (1.5) is

$$\eta = \tfrac{-(1+\xi_0\bar{\xi}_0)^2}{((1-\xi_0\bar{\xi}_0)\bar{\xi}_1-2\bar{\xi}_0)^2}\bar{\eta}_1 + \tfrac{2(\bar{\xi}_0-\bar{\xi}_1)(1+\xi_0\bar{\xi}_1)(1+\xi_0\bar{\xi}_0)}{((1-\xi_0\bar{\xi}_0)\bar{\xi}_1-2\bar{\xi}_0)^2}r_0. \tag{1.7}$$

The geometric content of this is: reflection of an oriented line can be decomposed into a sum of rotation about the origin (the derived action of PSL(2,$\mathbb{C}$) on T$\mathbb{P}^1$) and translation (a fibre-mapping on T$\mathbb{P}^1$).

2. Reflection off a translation invariant surface

Consider a translation invariant surface with axis lying along the x^3−axis in $\mathbb{R}^3$. Such a cylinder can be parametrised by $(u, v) \mapsto (z = z_0(u), t = v)$ for $(u, v) \in \mathbb{R}^2$, where $z = x^1 + ix^2$, $t = x^3$ and (x^1, x^2, x^3) are Euclidean coordinates in $\mathbb{R}^3$.

Proposition 1. *The normal congruence of a translation invariant surface is:*

$$\xi_0 = \pm\left[-\frac{\dot{z}_0}{\dot{\bar{z}}_0}\right]^{\frac{1}{2}} \tag{2.1}$$

$$\eta_0 = \tfrac{1}{2}(z_0 - 2v\xi_0 - \bar{z}_0\xi_0^2) \qquad\qquad r_0 = \frac{\bar{\xi}_0 z_0 + \xi_0\bar{z}_0 + (1-\xi_0\bar{\xi}_0)v}{1+\xi_0\bar{\xi}_0}, \tag{2.2}$$

where a dot represents differentiation with respect to u and the choice of sign of ξ_0 is one of orientation.

Proof. Let $\xi_0 \in S^2$ be the direction of the normal. This corresponds to the vector

$$\frac{2\xi_0}{1+\xi_0\bar{\xi}_0}\frac{\partial}{\partial z} + \frac{2\bar{\xi}_0}{1+\xi_0\bar{\xi}_0}\frac{\partial}{\partial \bar{z}} + \frac{1-\xi_0\bar{\xi}_0}{1+\xi_0\bar{\xi}_0}\frac{\partial}{\partial t}$$

The vanishing of the inner product of this vector with the push forward of $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ yields

$$1-\xi_0\bar{\xi}_0=0 \qquad\qquad \dot{z}_0\bar{\xi}_0+\dot{\bar{z}}_0\xi_0=0$$

and equations (2.1) follows.

Equations (2.2) come from inverting equation (1.1) for η and r. □

We consider the reflection of a point source off a surface that is translation invariant along the x^3-axis. By a translation of surface and source we move the point source to the origin. The line congruence consisting of all oriented lines through the origin is given by $\eta_1=0$ and $\xi_1\in S^2$.

Proposition 2. *The reflection of a point source at the origin off a translation invariant surface is* $\xi=-\xi_0^2\bar{\xi}_1$ *and* $\eta=\left(\bar{\xi}_0-\xi_0\bar{\xi}_1^2\right)\xi_0^2r_0$ *where*

$$\xi_1=\frac{u\pm(v^2+z_0\bar{z}_0)^{\frac{1}{2}}}{\bar{z}_0} \tag{2.3}$$

and ξ_0 *is given by (2.1). Here the* $\pm$ *refers to the two oriented lines from the source that intersect any given point in* $\mathbb{R}^3$*, and is chosen so that the ray goes from the source to the point of reflection.*

Proof. The reflection equations contained in Theorem 2, with $\eta_1=0$, $\xi_1\in S^2$ yield the stated reflected line congruence and the intersection equation has solution (2.3). □

Main Theorem:

The focal set generated by the reflection of a point source off a translation invariant surface consists of two sets: a curve and a surface. The focal curve lies in the plane orthogonal to the symmetry direction containing the source, while the focal surface is translation invariant.

Proof. The local parameter we choose is $\mu=u+iv$ and the spin coefficients are computed by inserting the reflected line congruence in Proposition 2 into equation (1.2). The focal set is then determined by inserting the reflected line congruence and the solutions r of the quadratic equation in Theorem 1 into (1.1). Assuming that both $\pm$ in (2.1) and (2.3) are positive, we obtain the following focal set:

$$z=\frac{z_0\dot{\bar{z}}_0-\bar{z}_0\dot{z}_0}{\dot{\bar{z}}_0} \qquad\qquad t=0,$$

and

$$z=\frac{2\ddot{z}_0\dot{\bar{z}}_0z_0^2\bar{z}_0-2\ddot{\bar{z}}_0\dot{z}_0z_0^2\bar{z}_0+\dot{z}_0^3\bar{z}_0^2-2\dot{z}_0^2\dot{\bar{z}}_0z_0\bar{z}_0+\dot{z}_0\dot{\bar{z}}_0^2z_0^2}{2\ddot{z}_0\dot{\bar{z}}_0z_0\bar{z}_0-2\ddot{\bar{z}}_0\dot{z}_0z_0\bar{z}_0-\dot{z}_0^2\dot{\bar{z}}_0\bar{z}_0+\dot{z}_0\dot{\bar{z}}_0^2z_0},$$

$$t=\frac{2v\;z_0\bar{z}_0(\ddot{\bar{z}}_0\dot{z}_0-\ddot{z}_0\dot{\bar{z}}_0)}{2\ddot{z}_0\dot{\bar{z}}_0z_0\bar{z}_0-2\ddot{\bar{z}}_0\dot{z}_0z_0\bar{z}_0-\dot{z}_0^2\dot{\bar{z}}_0\bar{z}_0+\dot{z}_0\dot{\bar{z}}_0^2z_0}.$$

The first of these is a curve in the x^1x^2-plane parameterised by u, and the second is a surface that is invariant in the x^3-direction.

A similar result holds for the other choices of orientation. □

References

[1] V. I. Arnold, S. M. Gusein-Zade, A. N. Varchenko, Singularities of differentiable maps, Volume 1, Birkhaeuser, Basel, 1986.

[2] J. Bruce, P. Giblin and C. Gibson, *On caustics by reflection*, Topology **21** (1982), 179-199.

[3] B. Guilfoyle and W. Klingenberg, *Generalised surfaces in* $\mathbb{R}^3$, Math. Proc. R. Ir. Acad. **104A(2)**, 199-209.

[4] B. Guilfoyle and W. Klingenberg, *Reflection of a wave off a surface*, Journal of Geometry (to appear) [math.DG/0406212]

[5] S. Izumiya, K. Saji and N. Takeuchi, *Singularities of line congruences*, Proceedings of the Royal Society of Edinburgh **133A** (2003), 1341-1359.

[6] R. Penrose and W. Rindler, Spinors and spacetime, Volume 1 and 2, Cambridge University Press, Cambridge, 1986.

Brendan Guilfoyle, Department of Mathematics and Computing, Institute of Technology, Tralee, Clash, Tralee, Co. Kerry, Ireland.

E-mail address: `brendan.guilfoyle@ittralee.ie`

Wilhelm Klingenberg, Department of Mathematical Sciences, University of Durham, Durham DH1 3LE, United Kingdom.

E-mail address: `wilhelm.klingenberg@durham.ac.uk`