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A CHARACTERIZATION OF WEINGARTEN SURFACES IN HYPERBOLIC 3-SPACE

NIKOS GEORGIOU AND BRENDAN GUILFOYLE

ABSTRACT. We study 2-dimensional submanifolds of the space $\mathbb{L}(\mathbb{H}^3)$ of oriented geodesics of hyperbolic 3-space, endowed with the canonical neutral Kähler structure. Such a surface is Lagrangian iff there exists a surface in \mathbb{H}^3 orthogonal to the geodesics of Σ .

We prove that the induced metric on a Lagrangian surface in $\mathbb{L}(\mathbb{H}^3)$ has zero Gauss curvature iff the orthogonal surfaces in \mathbb{H}^3 are Weingarten: the eigenvalues of the second fundamental form are functionally related. We then classify the totally null surfaces in $\mathbb{L}(\mathbb{H}^3)$ and recover the well-known holomorphic constructions of flat and CMC 1 surfaces in \mathbb{H}^3 .

Recently the existence and uniqueness of a canonical neutral Kähler structure on the space $\mathbb{L}(\mathbb{H}^3)$ of oriented geodesics of hyperbolic 3-space \mathbb{H}^3 has been established [4] [12]. The main purpose of this paper is to apply this work to the study of surfaces S in \mathbb{H}^3 . The oriented geodesics normal to S form a surface in $\mathbb{L}(\mathbb{H}^3)$ which is Lagrangian with respect to this Kähler structure. In fact, the study of surfaces in \mathbb{H}^3 is equivalent, at least locally, to the study of Lagrangian surfaces in $\mathbb{L}(\mathbb{H}^3)$.

Special classes of surfaces in hyperbolic 3-space have been studied for many decades, with various constructions being developed and applied to the particular class of surfaces under consideration. For example, surfaces of constant mean curvature 1 have been investigated in [1], while flat surfaces have been treated in [7] [8] [11]. The common feature of these surfaces is that they are Weingarten [14]: they have some specified functional relationship between the eigenvalues of the second fundamental form. Other forms of this relationship have been considered [3] [9] and uniqueness results obtained for Weingarten surfaces in \mathbb{R}^3 [2] [6].

In this paper we give a new characterization of the Weingarten condition for surfaces in \mathbb{H}^3 :

Main Theorem:

Let $S \subset \mathbb{H}^3$ be a C^4 smooth immersed oriented surface and $\Sigma \subset \mathbb{L}(\mathbb{H}^3)$ be the Lagrangian surface formed by the oriented geodesics normal to S . Assume that the metric \mathbb{G}_Σ induced on Σ by the neutral Kähler metric is non-degenerate. Then S is Weingarten iff the Gauss curvature of \mathbb{G}_Σ is zero.

The proof of this result follows from a careful study of 2-dimensional submanifolds of $\mathbb{L}(\mathbb{H}^3)$ using local coordinates, moving frames and the correspondence space.

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In the following section we give the geometric background (further details can be found in [4]) and explore the submanifold theory of surfaces in $\mathbb{L}(\mathbb{H}^3)$. Section three contains the proof of the Main Theorem.

In the final section we classify the totally null surfaces in $\mathbb{L}(\mathbb{H}^3)$ and recover the holomorphic constructions of flat and CMC 1 surfaces in \mathbb{H}^3 .

1. GEOMETRIC BACKGROUND

We briefly recall the basic construction of the canonical neutral Kähler metric on the space $\mathbb{L}(\mathbb{H}^3)$ of oriented geodesics of \mathbb{H}^3 - further details can be found in [4]. We use one of two models of \mathbb{H}^3 , the Poincaré ball model:

$$B^3 = \{(y^1, y^2, y^3) \in \mathbb{R}^3 \mid (y^1)^2 + (y^2)^2 + (y^3)^2 < 1\},$$

with standard coordinates (y^1, y^2, y^3) on \mathbb{R}^3 , and hyperbolic metric

$$ds^2 = \frac{4[(dy^1)^2 + (dy^2)^2 + (dy^3)^2]}{[1 - (y^1)^2 - (y^2)^2 - (y^3)^2]^2}.$$

and the upper-half space model:

$$\mathbb{R}_+^3 = \{(x^0, x^1, x^2) \in \mathbb{R}^3 \mid x^0 > 0\},$$

for standard coordinates (x^0, x^1, x^2) on \mathbb{R}^3 . In these coordinates the hyperbolic metric has expression:

$$d\bar{s}^2 = \frac{(dx^0)^2 + (dx^1)^2 + (dx^2)^2}{(x^0)^2}.$$

These are related by the mapping $\mathbb{R}_+^3 \rightarrow B^3: (x^0, x^1, x^2) \mapsto (y^1, y^2, y^3)$ defined by

$$y^1 = \frac{2x^1}{(x^0 + 1)^2 + (x^1)^2 + (x^2)^2}, \quad y^2 = \frac{2x^2}{(x^0 + 1)^2 + (x^1)^2 + (x^2)^2},$$

$$y^3 = \frac{(x^0)^2 + (x^1)^2 + (x^2)^2 - 1}{(x^0 + 1)^2 + (x^1)^2 + (x^2)^2}.$$

An oriented geodesic in \mathbb{H}^3 is uniquely determined by its beginning and end point on the boundary of the ball model, and so $\mathbb{L}(\mathbb{H}^3)$ can be identified with $S^2 \times S^2 - \Delta$, where Δ is the diagonal in $S^2 \times S^2$. Endowing $S^2 \times S^2 - \Delta$ with the standard differentiable structure, a tangent vector to an oriented geodesic $\gamma \in \mathbb{L}(\mathbb{H}^3)$ can then be identified with an orthogonal Jacobi field along $\gamma \subset \mathbb{H}^3$.

Rotation of Jacobi fields through 90° about γ defines an almost complex structure on $\mathbb{L}(\mathbb{H}^3)$. This almost complex structure is integrable, and so $\mathbb{L}(\mathbb{H}^3)$ becomes a complex surface, which turns out to be biholomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 - \bar{\Delta}$. Here $\bar{\Delta}$ is the “reflected” diagonal: in terms of holomorphic coordinates (μ_1, μ_2) on $\mathbb{P}^1 \times \mathbb{P}^1$, $\bar{\Delta} = \{(\mu_1, \mu_2) : \mu_1 \bar{\mu}_2 = -1\}$. This distinction between $\mathbb{P}^1 \times \mathbb{P}^1 - \Delta$ and $\mathbb{P}^1 \times \mathbb{P}^1 - \bar{\Delta}$ is crucial, as explained in section 4.4.

The complex structure \mathbb{J} on $\mathbb{L}(\mathbb{H}^3)$ can be supplemented with a compatible symplectic structure Ω , which has the following expression in holomorphic coordinates:

$$\Omega = - \left[\frac{1}{(1 + \mu_1 \bar{\mu}_2)^2} d\mu_1 \wedge d\bar{\mu}_2 + \frac{1}{(1 + \bar{\mu}_1 \mu_2)^2} d\bar{\mu}_1 \wedge d\mu_2 \right]. \quad (1.1)$$

Together we obtain a Kähler metric $\mathbb{G}(\cdot, \cdot) = \Omega(\mathbb{J}\cdot, \cdot)$:

$$\mathbb{G} = -i \left[\frac{1}{(1 + \mu_1 \bar{\mu}_2)^2} d\mu_1 \otimes d\bar{\mu}_2 - \frac{1}{(1 + \bar{\mu}_1 \mu_2)^2} d\bar{\mu}_1 \otimes d\mu_2 \right]. \quad (1.2)$$

This metric, which has signature $++--$, is invariant under the action induced on $\mathbb{L}(\mathbb{H}^3)$ by the isometry group of \mathbb{H}^3 . Indeed, this has been shown to be the unique Kähler metric on $\mathbb{L}(\mathbb{H}^3)$ with this property [12].

In order to transfer geometric data between $\mathbb{L}(\mathbb{H}^3)$ and \mathbb{H}^3 we use a correspondence space:

$$\begin{array}{ccc} \mathbb{L}(\mathbb{H}^3) \times \mathbb{R} & & \\ \pi_1 \downarrow & \searrow \Phi & \\ \mathbb{L}(\mathbb{H}^3) & & \mathbb{H}^3 \end{array}$$

The key property of this correspondence is that, given $\gamma \in \mathbb{L}(\mathbb{H}^3)$, the set $\Phi \circ \pi_1^{-1}(\gamma)$ is the oriented geodesic in \mathbb{H}^3 , while, for a point $p \in \mathbb{H}^3$, $\pi_1 \circ \Phi^{-1}(p)$ is the set of oriented geodesics in $\mathbb{L}(\mathbb{H}^3)$ that pass through p .

The map Φ takes an oriented geodesic γ in $\mathbb{L}(\mathbb{H}^3)$ and a real number r to the point on γ an affine parameter distance r from some fixed point on the geodesic. This choice of point on each geodesic can be made globally, but we more often just use a local choice, which is sufficient for our purposes.

In terms of holomorphic coordinates (μ_1, μ_2) on $\mathbb{L}(\mathbb{H}^3)$ and upper-half space coordinates (x^0, x^1, x^2) the map Φ has expression:

$$z = \frac{1 - \mu_1 \bar{\mu}_2}{2\bar{\mu}_2} + \left(\frac{1 + \mu_1 \bar{\mu}_2}{2\bar{\mu}_2} \right) \tanh r, \quad t = \frac{|1 + \bar{\mu}_1 \mu_2|}{2|\mu_2| \cosh r}.$$

where $z = x^1 + ix^2$ and $t = x^0$.

We often use local coordinates (ξ, η) on $\mathbb{L}(\mathbb{H}^3)$ which are related to the holomorphic coordinates by

$$\xi = \frac{2\mu_2}{1 + \bar{\mu}_1 \mu_2}, \quad \eta = \frac{1 - \mu_1 \bar{\mu}_2}{2\bar{\mu}_2}. \quad (1.3)$$

The resulting expression for Φ is

$$z = \eta + \frac{\tanh r}{\bar{\xi}}, \quad t = \frac{1}{|\xi| \cosh r}. \quad (1.4)$$

2. SURFACES IN $\mathbb{L}(\mathbb{H}^3)$

In this section we investigate *oriented geodesic congruences*, that is, surfaces $\Sigma \subset \mathbb{L}(\mathbb{H}^3)$, or two parameter families of oriented geodesics in \mathbb{H}^3 .

2.1. Coordinates and an adapted frame in \mathbb{H}^3 . Let (x^0, x^1, x^2) be the standard coordinates on the upper half space model of \mathbb{H}^3 with $x^0 > 0$ and set $z = x^1 + ix^2$, $\bar{z} = x^1 - ix^2$ and $t = x^0$.

Definition 1. A *null frame* in \mathbb{H}^3 is a trio $\{e_{(0)}, e_{(+)}, e_{(-)}\}$ of complex vector fields in $\mathbb{C} \otimes \mathbb{T}\mathbb{H}^3$ where $e_{(0)}$ is real, $e_{(+)}$ is the complex conjugate of $e_{(-)}$ and they satisfy the following properties:

$$\langle e_{(0)}, e_{(0)} \rangle = 1, \quad \langle e_{(0)}, e_{(+)} \rangle = \langle e_{(0)}, e_{(-)} \rangle = 0, \quad \langle e_{(+)}, e_{(-)} \rangle = 1,$$

where \langle, \rangle is the hyperbolic metric.

Given an orthonormal frame $\{e_0, e_1, e_2\}$ on $\mathbb{T}\mathbb{H}^3$ we construct a null frame by

$$e_{(+)} = \frac{1}{\sqrt{2}}(e_1 - ie_2), \quad e_{(-)} = \frac{1}{\sqrt{2}}(e_1 + ie_2), \quad e_{(0)} = e_0.$$

Definition 2. Given a surface $\Sigma \subset \mathbb{L}(\mathbb{H}^3)$ an *adapted null frame* is a null frame $\{e_{(0)}, e_{(+)}, e_{(-)}\}$ such that, for each $\gamma \in \Sigma$, we have $e_{(0)} = \dot{\gamma}$, and the orientation of $\{e_{(0)}, \operatorname{Re}(e_{(+)})\}$ is the standard orientation on \mathbb{H}^3 .

In what follows we consider a locally parameterized surface $\Sigma \subset \mathbb{L}(\mathbb{H}^3)$:

Proposition 1. Let $\Sigma \subset \mathbb{L}(\mathbb{H}^3)$ be a surface given locally by an immersion $f : \mathbb{C} \rightarrow \mathbb{L}(\mathbb{H}^3) : (\nu, \bar{\nu}) \mapsto (\xi(\nu, \bar{\nu}), \eta(\nu, \bar{\nu}))$. Then an adapted null frame $\{e_{(0)}, e_{(+)}, e_{(-)}\}$ is given by

$$e_{(0)} = \frac{\partial}{\partial r}, \quad e_{(+)} = a \frac{\partial}{\partial \nu} + b \frac{\partial}{\partial \bar{\nu}} + \omega \frac{\partial}{\partial r},$$

where

$$\omega = \sqrt{2} \Delta^{-1} \left(\overline{\partial^- F} (\bar{\xi} \bar{\partial} \eta + \xi \bar{\partial} \bar{\eta}) - \overline{\partial^+ F} (\bar{\xi} \partial \eta + \xi \partial \bar{\eta}) \right),$$

$$a = 2\sqrt{2} \Delta^{-1} \overline{\partial^+ F}, \quad b = -2\sqrt{2} \Delta^{-1} \overline{\partial^- F}, \quad \Delta = \partial^+ F \overline{\partial^+ F} - \partial^- F \overline{\partial^- F}, \quad (2.1)$$

and

$$\begin{aligned} \partial^+ F &= \xi e^r \partial \bar{\eta} - \bar{\xi} e^{-r} \partial \eta - e^r \partial \ln \xi - e^{-r} \partial \ln \bar{\xi}, \\ \partial^- F &= \xi e^r \bar{\partial} \bar{\eta} - \bar{\xi} e^{-r} \bar{\partial} \eta - e^r \bar{\partial} \ln \xi - e^{-r} \bar{\partial} \ln \bar{\xi}. \end{aligned} \quad (2.2)$$

Proof. A parameterized surface Σ gives (composing with the map Φ in (1.4)) a map

$$\mathbb{C} \times \mathbb{R} \rightarrow \mathbb{H}^3 : (\nu, \bar{\nu}, r) \mapsto (z(\nu, \bar{\nu}, r), \bar{z}(\nu, \bar{\nu}, r)). \quad (2.3)$$

Consider now the adapted null frame $\{e_{(0)}, e_{(+)}, e_{(-)}\}$ given by Proposition 4 of [4], the derivative of the map (2.3) gives

$$\frac{\partial}{\partial r} \mapsto e_{(0)},$$

and

$$\begin{aligned} \frac{\partial}{\partial \nu} &\mapsto \frac{1}{2} (\bar{\xi} \partial \eta + \xi \partial \bar{\eta}) e_{(0)} + \frac{1}{2\sqrt{2}} (\xi e^r \partial \bar{\eta} - \bar{\xi} e^{-r} \partial \eta - e^r \partial \ln \xi - e^{-r} \partial \ln \bar{\xi}) e_{(+)} \\ &\quad + \frac{1}{2\sqrt{2}} (\bar{\xi} e^r \partial \eta - \xi e^{-r} \partial \bar{\eta} - e^r \partial \ln \bar{\xi} - e^{-r} \partial \ln \xi) e_{(-)}, \end{aligned}$$

The inverse of the above transformation gives the adapted null frame. \square

2.2. The local geometry of geodesic congruences. We now investigate the local geometry of surfaces in $\mathbb{L}(\mathbb{H}^3)$. Suppose that Σ is such a surface and $\{e_{(0)}, e_{(+)}, e_{(-)}\}$ is an adapted null frame, and define the two complex scalars:

$$\rho = \langle \nabla_{e_{(-)}} e_{(+)}, e_{(0)} \rangle, \quad \sigma = \langle \nabla_{e_{(+)}} e_{(+)}, e_{(0)} \rangle.$$

The complex *optical scalar* functions ρ and σ describe the first order geometric behaviour of the surface. The real part of ρ is referred to as the *divergence*, the imaginary part is the *twist*, which is denoted by λ , and σ is the *shear* of the geodesics. They satisfy the Sachs equations for \mathbb{H}^3 [10]:

$$\frac{\partial \rho}{\partial r} = \rho^2 + \sigma \bar{\sigma} - 1, \quad \frac{\partial \sigma}{\partial r} = (\rho + \bar{\rho})\sigma,$$

where r is an affine parameter along the geodesics. Note that if the shear or the twist vanish at some point of the geodesic, they vanish at every point of the geodesic by iterations of the Sachs equations.

Proposition 2. *The optical scalars of a surface $(\nu, \bar{\nu}) \mapsto (\xi(\nu, \bar{\nu}), \eta(\nu, \bar{\nu}))$ are:*

$$\begin{aligned} \sigma &= \frac{2e^{-r}}{\Delta} \left[(\xi \bar{\partial} \bar{\eta} + \bar{\partial} \ln \xi) \overline{\partial^- F} - (\xi \partial \bar{\eta} + \partial \ln \xi) \overline{\partial^+ F} \right], \\ \rho &= -1 + \frac{2e^{-r}}{\Delta} \left[(\xi \partial \bar{\eta} + \partial \ln \xi) \partial^- F - (\xi \bar{\partial} \bar{\eta} + \bar{\partial} \ln \xi) \partial^+ F \right], \end{aligned}$$

where $\partial^\pm F$ and Δ are given by (2.1) and (2.2), and ∂ is differentiation with respect to ν .

Proof. The Levi-Civita connection of the hyperbolic metric is torsion-free and so we have the following expression for the Lie derivative of $e_{(+)}$ in the $e_{(0)}$ direction

$$L_{e_{(0)}} e_{(+)} = \bar{\rho} e_{(+)} + \sigma e_{(-)}.$$

Recall the expression for $e_{(+)}$ given in Proposition 1 and, since $\langle e_{(0)}, e_{(+)} \rangle = 0$, we find

$$\omega = -a \langle e_{(0)}, \frac{\partial}{\partial \nu} \rangle - b \langle e_{(0)}, \frac{\partial}{\partial \bar{\nu}} \rangle.$$

Therefore, $e_{(+)} = aZ_+ + bZ_-$ where

$$Z_+ = \frac{\partial}{\partial \nu} - \langle e_{(0)}, \frac{\partial}{\partial \nu} \rangle \frac{\partial}{\partial r}, \quad Z_- = \frac{\partial}{\partial \bar{\nu}} - \langle e_{(0)}, \frac{\partial}{\partial \bar{\nu}} \rangle \frac{\partial}{\partial r}.$$

After computing the hyperbolic metric in $(\nu, \bar{\nu}, r)$ coordinates we find that

$$\langle Z_+, Z_+ \rangle = \frac{1}{4} \partial^+ F \overline{\partial^- F}, \quad \langle Z_+, Z_- \rangle = \frac{1}{8} \left(\partial^+ F \overline{\partial^+ F} + \partial^- F \overline{\partial^- F} \right).$$

The $e_{(+)}$ component of $L_{e_{(0)}} e_{(+)}$ is $\sigma = \langle L_{e_{(0)}} e_{(+)}, e_{(+)} \rangle$ and so

$$\sigma = \frac{1}{2} \frac{\partial a^2}{\partial r} \langle Z_+, Z_+ \rangle + \frac{1}{2} \frac{\partial b^2}{\partial r} \langle Z_-, Z_- \rangle + \frac{\partial ab}{\partial r} \langle Z_+, Z_- \rangle.$$

After a lengthy computation we find that

$$\sigma = 2e^{-r} \frac{(\xi \bar{\partial} \bar{\eta} + \bar{\partial} \ln \xi) \overline{\partial^- F} - (\xi \partial \bar{\eta} + \partial \ln \xi) \overline{\partial^+ F}}{\partial^+ F \overline{\partial^+ F} - \partial^- F \overline{\partial^- F}}.$$

The scalar ρ can be found in a similar way. \square

It's useful to know ρ and σ in terms of the holomorphic coordinates (μ_1, μ_2) .

Proposition 3. *Let Σ be a geodesic congruence given by an immersion $f : \Sigma \rightarrow \mathbb{L}(\mathbb{H}^3) : (\nu, \bar{\nu}) \mapsto (\mu_1(\nu, \bar{\nu}), \mu_2(\nu, \bar{\nu}))$. The optical scalars ρ and σ are*

$$\sigma = \frac{8\mu_2 J_{\bar{2}\bar{1}}}{\bar{\mu}_2 \Delta |1 + \mu_1 \bar{\mu}_2|^2}, \quad \rho = -1 - \frac{8e^{-r}}{\Delta} \left[\frac{J_{2\bar{1}}}{(1 + \bar{\mu}_1 \mu_2)^2} e^r - \frac{|\mu_2|^2 J_{1\bar{1}}}{|1 + \bar{\mu}_1 \mu_2|^2} e^{-r} \right], \quad (2.4)$$

where

$$J_{kl} = \partial \mu_k \bar{\partial} \mu_l - \bar{\partial} \mu_k \partial \mu_l \quad k, l = 1, 2, \bar{1}, \bar{2},$$

and

$$\frac{1}{4} \Delta = \frac{J_{2\bar{2}}}{|\mu_2|^2 |1 + \mu_1 \bar{\mu}_2|^2} e^{2r} + \frac{J_{\bar{2}1}}{(1 + \mu_1 \bar{\mu}_2)^2} + \frac{J_{1\bar{2}}}{(1 + \bar{\mu}_1 \mu_2)^2} + \frac{|\mu_2|^2 J_{1\bar{1}}}{|1 + \mu_1 \bar{\mu}_2|^2} e^{-2r}.$$

Proof. This follows from the expressions in Proposition 3 and the transformation (1.3). \square

2.3. Lagrangian Surfaces.

Definition 3. A point γ on a surface $\Sigma \subset \mathbb{L}(\mathbb{H}^3)$ is said to be a *Lagrangian point* if $f^* \Omega = 0$ at γ , where Ω is the symplectic 2-form on $\mathbb{L}(\mathbb{H}^3)$. A surface $\Sigma \subset \mathbb{L}(\mathbb{H}^3)$ is said to be *Lagrangian* if all of the points of Σ are Lagrangian points.

Proposition 4. *A surface Σ is Lagrangian iff the imaginary part of ρ (the twist) is zero.*

Proof. Let $f : \Sigma \rightarrow \mathbb{L}(\mathbb{H}^3) : (\nu, \bar{\nu}) \mapsto (\mu_1(\nu, \bar{\nu}), \mu_2(\nu, \bar{\nu}))$ be the C^2 immersion of the geodesic congruence.

The twist $\lambda = \text{Im} \rho$ is

$$\lambda = \frac{4i}{\Delta} \left[\frac{J_{2\bar{1}}}{(1 + \bar{\mu}_1 \mu_2)^2} - \frac{J_{1\bar{2}}}{(1 + \mu_1 \bar{\mu}_2)^2} \right].$$

The symplectic 2-form Ω in $\mathbb{L}(\mathbb{H}^3)$ is given by (1.1) and so $f^* \Omega$ is

$$f^* \Omega = \left[\frac{J_{2\bar{1}}}{(1 + \bar{\mu}_1 \mu_2)^2} - \frac{J_{1\bar{2}}}{(1 + \mu_1 \bar{\mu}_2)^2} \right] d\nu \wedge d\bar{\nu} = \frac{\Delta}{4i} \lambda d\nu \wedge d\bar{\nu}.$$

The proposition then follows. \square

Definition 4. An oriented geodesic congruence is *integrable* iff locally there exists an embedded surface S in \mathbb{H}^3 such that S is orthogonal to the geodesics of the congruence.

Proposition 5. *A geodesic congruence Σ is integrable iff Σ is Lagrangian.*

Proof. The geodesics are integrable iff, in an adapted frame, $[e_{(+)}, e_{(-)}]$ lies in the span of $\{e_{(+)}, e_{(-)}\}$ (by Frobenius' theorem). But, using the fact that the Levi-Civita connection of the hyperbolic metric is torsion free,

$$\langle [e_{(+)}, e_{(-)}], e_{(0)} \rangle = \langle \nabla_{e_{(+)}} e_{(-)}, e_{(0)} \rangle - \langle \nabla_{e_{(-)}} e_{(+)}, e_{(0)} \rangle = \bar{\rho} - \rho = -2\lambda i$$

Thus integrability is equivalent to the imaginary part of ρ vanishing, as claimed. \square

Proposition 6. *Let Σ be a Lagrangian surface in $\mathbb{L}(\mathbb{H}^3)$ parameterized by $\nu \rightarrow (\mu_1(\nu, \bar{\nu}), \mu_2(\nu, \bar{\nu}))$. Then the surfaces $S \subset \mathbb{H}^3$ orthogonal to the geodesics of Σ are given by the functions $r = r(\nu, \bar{\nu})$ which solve:*

$$2\partial r = \frac{\mu_2}{\bar{\mu}_1\mu_2 + 1} \left(\partial\bar{\mu}_1 + \frac{\partial\mu_2}{\mu_2^2} \right) + \frac{\bar{\mu}_2}{\mu_1\bar{\mu}_2 + 1} \left(\partial\mu_1 + \frac{\partial\bar{\mu}_2}{\bar{\mu}_2^2} \right).$$

Here the real constant of integration parameterizes the family of parallel surfaces orthogonal to the geodesics.

Proof. A straightforward computation shows that the parametric surface in \mathbb{H}^3 , obtained by inserting $r = r(\nu, \bar{\nu})$ in (1.4), is orthogonal to the congruence iff the above condition holds. \square

Definition 5. Given an immersion $f : \Sigma \rightarrow \mathbb{L}(\mathbb{H}^3)$, consider the map $(\pi \circ f)_* : T\Sigma \rightarrow T\mathbb{P}^1$, where π is projection onto the first factor of $\mathbb{L}(\mathbb{H}^3) = \mathbb{P}^1 \times \mathbb{P}^1 - \bar{\Delta}$. The *rank* of the immersion f at a point $\gamma \in \Sigma$ is defined to be the rank of this map at γ , which can be 0, 1 or 2.

Note 1. Alternatively, the rank could be defined by projection onto the second factor. However, by reversing the orientation of the geodesics in a geodesic congruence the two definitions would switch. In what follows we take projection onto the first factor.

A rank 0 Lagrangian surface is a horosphere (see Definition 10).

A rank 2 surface Σ , parameterized by $\mu_2 = \mu_2(\mu_1, \bar{\mu}_1)$, is Lagrangian iff there is a smooth real function r such that

$$2\partial_1 r = \frac{\partial_1 \mu_2}{\mu_2(1 + \bar{\mu}_1 \mu_2)} + \frac{\partial_1 \bar{\mu}_2}{\bar{\mu}_2(1 + \mu_1 \bar{\mu}_2)} + \frac{\bar{\mu}_2}{1 + \mu_1 \bar{\mu}_2}, \quad (2.5)$$

where here ∂_1 is differentiation with respect to μ_1 .

In the Lagrangian case, the functions σ and ρ have the following interpretation in terms of the second fundamental form of the orthogonal surfaces in \mathbb{H}^3 .

Proposition 7. *Let $S \subset \mathbb{H}^3$ be a C^2 immersed surface and $\Sigma \subset \mathbb{L}(\mathbb{H}^3)$ be the oriented normal geodesics. Then*

$$|\sigma| = \frac{1}{2}|\lambda_1 - \lambda_2| \quad \rho = -\frac{1}{2}(\lambda_1 + \lambda_2), \quad (2.6)$$

where λ_1 and λ_2 are the principal curvatures of S .

Proof. Let S be a C^2 surface immersed in \mathbb{H}^3 and let N_i be the unit normal 1-form. The second fundamental form is a symmetric two tensor on S defined by

$$H_{ij} = P_i^k P_j^l \nabla_k N_l,$$

where ∇ is the Levi-Civita connection on \mathbb{H}^3 and P_i^j is orthogonal projection onto the tangent space of S . Such a symmetric two tensor has two real eigenvalues, λ_1 and λ_2 , at each point of S . These are called the *principal curvatures* of S and the associated eigen-directions are called *principal directions* for the surface. Let $\{e_{(1)}, e_{(2)}\}$ be an orthonormal eigen-basis for H_{ij} , and then the null frame is

$e_{(+)} = \frac{1}{\sqrt{2}}(e_{(1)} + ie_{(2)})e^{i\alpha}$ for some angle α . Computing the null frame components of H_{ij}

$$H_{(++)} = \langle \nabla_{e_{(+)}} \theta^{(0)}, e_{(+)} \rangle = \Gamma_{(+)}^{(0)} = \sigma,$$

$$H_{(+-)} = \langle \nabla_{e_{(+)}} \theta^{(0)}, e_{(-)} \rangle = \Gamma_{(+)}^{(0)} = -\bar{\rho} = -\rho.$$

In terms of the real basis

$$\sigma = H_{(++)} = \frac{1}{2}(H_{(11)} - H_{(22)})e^{2i\alpha} = \frac{1}{2}(\lambda_1 - \lambda_2)e^{2i\alpha},$$

$$\rho = -H_{(+-)} = -\frac{1}{2}(H_{(11)} + H_{(22)}) = -\frac{1}{2}(\lambda_1 + \lambda_2),$$

as claimed. \square

The Gauss curvature of S is $\kappa = \rho\bar{\rho} - \sigma\bar{\sigma} - 1$, and in terms of local coordinates $(\nu, \bar{\nu})$ this is:

$$\kappa = \frac{8}{\Delta} \left[\frac{J_{2\bar{1}}}{(1 + \bar{\mu}_1\mu_2)^2} + \frac{J_{1\bar{2}}}{(1 + \mu_1\bar{\mu}_2)^2} \right]. \quad (2.7)$$

2.4. Complex Curves.

Definition 6. A point γ on a surface $\Sigma \subset \mathbb{L}(\mathbb{H}^3)$ is said to be a *complex point* if the complex structure \mathbb{J} acting on $\mathbb{L}(\mathbb{H}^3)$ preserves $T_\gamma\Sigma$. A surface $\Sigma \subset \mathbb{L}(\mathbb{H}^3)$ is said to be a *complex curve* if all of the points of Σ are complex points.

In particular:

Proposition 8. *A point γ on a surface Σ is complex iff the shear vanishes along γ .*

Proof. Let $f: \Sigma \rightarrow \mathbb{L}(\mathbb{H}^3): (\nu, \bar{\nu}) \mapsto (\mu_1(\nu, \bar{\nu}), \mu_2(\nu, \bar{\nu}))$ be the smooth immersion of Σ in $\mathbb{L}(\mathbb{H}^3)$. We consider the derivative of the immersion $df: T\Sigma \rightarrow T\mathbb{L}(\mathbb{H}^3)$. Let j be a conformal structure on Σ compatible with ν . We define the sections $\delta^+f \in \Lambda^{10}(\Sigma) \otimes T^{10}\mathbb{L}(\mathbb{H}^3)$ and $\delta^-f \in \Lambda^{01}(\Sigma) \otimes T^{10}\mathbb{L}(\mathbb{H}^3)$ by

$$\delta^+f = \frac{1}{2}(df - \mathbb{J} \circ df \circ j) \quad \delta^-f = \frac{1}{2}(df + \mathbb{J} \circ df \circ j).$$

Then $\delta^+f \wedge \delta^-f \in \Lambda^2(\Sigma) \otimes \det T^{10}\mathbb{L}(\mathbb{H}^3)$ works out to be

$$\delta^+f \wedge \delta^-f = J_{12}d\nu \wedge d\bar{\nu}.$$

Now, a point is complex iff this 2-form vanishes. On the other hand we have

$$\sigma = \frac{8\mu_2 J_{2\bar{1}}}{\bar{\mu}_2 \Delta |1 + \mu_1 \bar{\mu}_2|^2} = \frac{8\mu_2 \overline{J_{12}}}{\bar{\mu}_2 \Delta |1 + \mu_1 \bar{\mu}_2|^2},$$

and hence the proposition follows. \square

2.5. The Induced Metric.

Theorem 1. *Let Σ be a surface in $\mathbb{L}(\mathbb{H}^3)$. The induced metric is Lorentz (degenerate, Riemannian) iff $|\sigma|^2 - \lambda^2 > 0$ ($= 0, < 0$), where λ and σ are the twist and the shear of the geodesic congruence Σ .*

Proof. Let f be the smooth immersion like before with local complex coordinate ν . We have seen that the twist and the shear are given by

$$\sigma = \frac{8\mu_2 J_{2\bar{1}}}{\bar{\mu}_2 \Delta |1 + \mu_1 \bar{\mu}_2|^2} \quad \lambda = \frac{4i}{\Delta} \left[\frac{J_{2\bar{1}}}{(1 + \bar{\mu}_1 \mu_2)^2} - \frac{J_{1\bar{2}}}{(1 + \mu_1 \bar{\mu}_2)^2} \right].$$

Therefore we obtain

$$|\sigma|^2 - \lambda^2 = \frac{16}{\Delta^2} \left[\frac{J_{2\bar{1}}^2}{(1 + \bar{\mu}_1 \mu_2)^4} + \frac{J_{1\bar{2}}^2}{(1 + \mu_1 \bar{\mu}_2)^4} + \frac{4}{|1 + \mu_1 \bar{\mu}_2|^4} \left(J_{12} J_{2\bar{1}} - \frac{1}{2} J_{2\bar{1}} J_{1\bar{2}} \right) \right].$$

Pulling back the metric \mathbb{G} , which is given by (1.2), to Σ and computing the determinant

$$\det[f^* \mathbb{G}] = -\frac{\Delta^2}{64} (|\sigma|^2 - \lambda^2),$$

and the theorem follows. \square

We have the following corollary:

Corollary 1. *The induced metric on a Lagrangian surface in $\mathbb{L}(\mathbb{H}^3)$ is either Lorentz or degenerate, the latter occurring at umbilic points on the orthogonal surfaces in \mathbb{H}^3 .*

Proof. By Proposition 4 a Lagrangian surface has $\lambda = 0$, and so, by the preceding theorem, Σ is Lorentz or degenerate, according to whether σ is non-zero or zero. By Proposition 7 a point on the orthogonal surface at which $\sigma = 0$ is an umbilic point: $\lambda_1 = \lambda_2$. The result follows. \square

Similarly, we have:

Corollary 2. *The induced metric on a complex curve Σ in $\mathbb{L}(\mathbb{H}^3)$ is either Riemannian or degenerate, the latter occurring at Lagrangian points on Σ .*

3. WEINGARTEN SURFACES IN \mathbb{H}^3

Definition 7. A surface S in \mathbb{H}^3 is said to be *Weingarten* if the eigenvalues λ_1, λ_2 of the second fundamental form of S are functionally related [14]:

$$d\lambda_1 \wedge d\lambda_2 = 0.$$

we now prove our main result, namely:

Main Theorem:

Let $S \subset \mathbb{H}^3$ be a C^4 smooth immersed oriented surface and $\Sigma \subset \mathbb{L}(\mathbb{H}^3)$ be the Lagrangian surface formed by the oriented geodesics normal to S . Assume that the metric \mathbb{G}_Σ induced on Σ by the neutral Kähler metric is non-degenerate. Then S is Weingarten iff the Gauss curvature of \mathbb{G}_Σ is zero.

Proof. First assume that the normal geodesic congruence Σ is of rank 2 and so is locally parameterized by $\mu_1 \mapsto (\mu_1, \mu_2(\mu_1, \bar{\mu}_1))$. Our first task is to find the Gauss curvature K of the geodesic congruence. The induced metric g , computed by pulling back (1.2) to Σ , is:

$$g = \mathbb{G}_\Sigma = 2 \operatorname{Im} \left[\frac{\partial \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} d\mu_1^2 + \frac{\bar{\partial} \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} d\mu_1 d\bar{\mu}_1 \right], \quad (3.1)$$

where $\partial, \bar{\partial}$ are differentiation with respect of $\mu_1, \bar{\mu}_1$ respectively. By Proposition 4 the Lagrangian condition is $\operatorname{Im} \rho = 0$ and so by equation (2.4)

$$\frac{\bar{\partial} \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} - \frac{\partial \mu_2}{(1 + \bar{\mu}_1 \mu_2)^2} = 0. \quad (3.2)$$

Thus the induced metric on a Lagrangian surface simplifies to

$$g = -i \left[\frac{\partial \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} d\mu_1^2 - \frac{\bar{\partial} \mu_2}{(1 + \bar{\mu}_1 \mu_2)^2} d\bar{\mu}_1^2 \right].$$

The Christoffel symbols Γ_{ij}^k of the metric \mathbb{G}_Σ are

$$\begin{aligned} \Gamma_{11}^1 &= \frac{1}{2} \partial \left[\ln \left(\frac{\partial \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} \right) \right], & \Gamma_{\bar{1}\bar{1}}^1 &= \frac{1}{2} \bar{\partial} \left[\ln \left(\frac{\partial \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} \right) \right], \\ \Gamma_{\bar{1}\bar{1}}^{\bar{1}} &= \frac{(1 + \mu_1 \bar{\mu}_2)^2}{2 \partial \bar{\mu}_2} \partial \left[\left(\frac{\bar{\partial} \mu_2}{(1 + \bar{\mu}_1 \mu_2)^2} \right) \right]. \end{aligned}$$

Set

$$\sigma_0 = \frac{\partial \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2}, \quad (3.3)$$

so that the induced metric on Σ is

$$g = -i(\sigma_0 d\mu_1^2 - \bar{\sigma}_0 d\bar{\mu}_1^2),$$

and the Christoffel symbols can be written

$$\Gamma_{11}^1 = \frac{\partial \sigma_0}{2\sigma_0}, \quad \Gamma_{\bar{1}\bar{1}}^1 = \frac{\bar{\partial} \sigma_0}{2\sigma_0}, \quad \Gamma_{\bar{1}\bar{1}}^{\bar{1}} = \frac{\partial \bar{\sigma}_0}{2\sigma_0}.$$

The only non vanishing components of the Riemann tensor turn out to be

$$R_{1\bar{1}\bar{1}1} = \frac{i}{2} (\bar{\partial}^2 \sigma_0 - \partial^2 \bar{\sigma}_0) + \frac{i}{4} \left[\frac{(\partial \bar{\sigma}_0)^2 - \bar{\partial} \bar{\sigma}_0 \bar{\partial} \sigma_0}{\bar{\sigma}_0} - \frac{(\bar{\partial} \sigma_0)^2 - \partial \sigma_0 \partial \bar{\sigma}_0}{\sigma_0} \right].$$

For a 2-dimensional surface, we have the following relation that gives the Gauss curvature K :

$$R_{abcd} = K(g_{ac}g_{bd} - g_{ad}g_{bc}).$$

Using the above relation we obtain

$$K = \frac{R_{1\bar{1}\bar{1}1}}{g_{11}g_{\bar{1}\bar{1}}} = \frac{1}{|\sigma_0|^2} R_{1\bar{1}\bar{1}1},$$

and so

$$K = \frac{i}{4|\sigma_0|^2} \left[2(\bar{\partial}^2 \sigma_0 - \partial^2 \bar{\sigma}_0) + \frac{(\partial \bar{\sigma}_0)^2 - \bar{\partial} \bar{\sigma}_0 \bar{\partial} \sigma_0}{\bar{\sigma}_0} - \frac{(\bar{\partial} \sigma_0)^2 - \partial \sigma_0 \partial \bar{\sigma}_0}{\sigma_0} \right]. \quad (3.4)$$

If $\sigma_0 = 0$ then the induced metric vanishes, and we therefore exclude this case and assume that $\sigma_0 \neq 0$.

Now set

$$\rho_0 = \frac{\partial\mu_2}{(1 + \bar{\mu}_1\mu_2)^2}. \quad (3.5)$$

Since Σ is Lagrangian, ρ_0 is real function. Differentiating (3.3) and (3.5), we obtain

$$\begin{aligned} \bar{\partial}\rho_0 &= \frac{\partial\bar{\partial}\mu_2}{(1 + \bar{\mu}_1\mu_2)^2} - \frac{2\bar{\mu}_1\partial\mu_2\bar{\partial}\mu_2}{(1 + \bar{\mu}_1\mu_2)^3} - \frac{2\mu_2\bar{\partial}\mu_2}{(1 + \bar{\mu}_1\mu_2)^3} \\ \partial\bar{\sigma}_0 &= \frac{\partial\bar{\partial}\mu_2}{(1 + \bar{\mu}_1\mu_2)^2} - \frac{2\bar{\mu}_1\partial\mu_2\bar{\partial}\mu_2}{(1 + \bar{\mu}_1\mu_2)^3}, \end{aligned}$$

and therefore we have established the identity:

$$\partial\bar{\sigma}_0 = \bar{\partial}\rho_0 + \frac{2\mu_2}{1 + \bar{\mu}_1\mu_2}\rho_0. \quad (3.6)$$

We have seen that the Gauss curvature (3.4) has a term with second order derivatives of σ_0 and $\bar{\sigma}_0$. We can reduce the order of the derivatives using (3.6):

$$\bar{\partial}^2\sigma_0 = \partial\bar{\partial}\rho_0 + \bar{\partial}\left(\frac{2\bar{\mu}_2}{1 + \mu_1\bar{\mu}_2}\rho_0\right) \quad \partial^2\bar{\sigma}_0 = \partial\bar{\partial}\rho_0 + \partial\left(\frac{2\mu_2}{1 + \bar{\mu}_1\mu_2}\rho_0\right),$$

and subtracting we obtain

$$\bar{\partial}^2\sigma_0 - \partial^2\bar{\sigma}_0 = 2\left(\frac{\bar{\mu}_2}{1 + \mu_1\bar{\mu}_2}\partial\bar{\sigma}_0 - \frac{\mu_2}{1 + \bar{\mu}_1\mu_2}\bar{\partial}\sigma_0\right).$$

We now substitute the above into the expression (3.4) and obtain the Gauss curvature of Σ :

$$K = \frac{i}{|\sigma_0|^2} \left[\frac{\bar{\mu}_2\partial\bar{\sigma}_0}{1 + \mu_1\bar{\mu}_2} - \frac{\mu_2\bar{\partial}\sigma_0}{1 + \bar{\mu}_1\mu_2} + \frac{1}{4} \left(\frac{(\partial\bar{\sigma}_0)^2 - \bar{\partial}\bar{\sigma}_0\bar{\partial}\sigma_0}{\bar{\sigma}_0} - \frac{(\bar{\partial}\sigma_0)^2 - \partial\sigma_0\partial\bar{\sigma}_0}{\sigma_0} \right) \right].$$

Denote the Gauss curvature of the orthogonal surface S in \mathbb{H}^3 by κ . If $\kappa = 0$ then we have from equation (2.7) in graph coordinates that $\partial\mu_2 = 0$. Then, by expression (3.5) we have that $\rho_0 = 0$ and from the identity (3.6) we conclude that $\partial\bar{\sigma}_0 = 0$. Substituting this in the expression above for K , we have $K = 0$ and the surface in $\mathbb{L}(\mathbb{H}^3)$ is scalar flat, as claimed.

Now assume that $\kappa \neq 0$. If λ_1, λ_2 are the eigenvalues of the 2nd fundamental form, then $\kappa = \lambda_1\lambda_2 - 1$. From (2.6) we obtain

$$d\left(\frac{|\sigma|^2}{\kappa^2}\right) \wedge d\left(\frac{\rho+1}{\kappa}\right) = \frac{(\lambda_1 - \lambda_2)(\lambda_1 - 1)(\lambda_2 - 1)}{2(\lambda_1\lambda_2 - 1)^4} d\lambda_1 \wedge d\lambda_2.$$

Observe that, on an open set,

$$d\left(\frac{|\sigma|^2}{\kappa^2}\right) \wedge d\left(\frac{\rho+1}{\kappa}\right) = 0 \quad \text{iff} \quad d\lambda_1 \wedge d\lambda_2 = 0,$$

and therefore a necessary and sufficient condition for the the surface S to be Weingarten is

$$d\left(\frac{|\sigma|^2}{\kappa^2}\right) \wedge d\left(\frac{\rho+1}{\kappa}\right) = 0.$$

We are ready now to prove our result. We need to show that

$$d\left(\frac{|\sigma|^2}{\kappa^2}\right) \wedge d\left(\frac{\rho+1}{\kappa}\right) \propto K d\mu_1 \wedge d\bar{\mu}_1.$$

Expressing the Gauss curvature κ of S , and ρ, σ in terms of ρ_0, σ_0

$$\kappa = \frac{16}{\Delta}\rho_0, \quad \rho = -1 - \frac{8}{\Delta} \left(\rho_0 - \frac{|\mu_2|^2}{|1 + \bar{\mu}_1\mu_2|^2} e^{-2r} \right), \quad \sigma = \frac{8\mu_2(1 + \mu_1\bar{\mu}_2)}{\bar{\mu}_2\Delta(1 + \bar{\mu}_1\mu_2)}\sigma_0,$$

then

$$\frac{|\sigma|^2}{\kappa^2} = \frac{|\sigma_0|^2}{4\rho_0^2}, \quad \frac{\rho + 1}{\kappa} = -\frac{1}{2} + \frac{1}{2} \frac{|\mu_2|^2}{|1 + \bar{\mu}_1\mu_2|^2} \frac{e^{-2r}}{\rho_0}.$$

Computing the derivatives of the above we obtain

$$\begin{aligned} d\left(\frac{\rho + 1}{\kappa}\right) &= -\frac{\rho_0^{-1}|\mu_2|^2 e^{-2r}}{2|1 + \bar{\mu}_1\mu_2|^2} \left[\left(\frac{\partial\rho_0}{\rho_0} + \frac{2\bar{\mu}_2}{1 + \mu_1\bar{\mu}_2} \right) d\mu_1 + \left(\frac{\bar{\partial}\rho_0}{\rho_0} + \frac{2\mu_2}{1 + \bar{\mu}_1\mu_2} \right) d\bar{\mu}_1 \right], \\ d\left(\frac{|\sigma|^2}{\kappa^2}\right) &= \left[\left(\partial|\sigma_0|^2 - 2|\sigma_0|^2 \frac{\partial\rho_0}{\rho_0} \right) d\mu_1 + \left(\bar{\partial}|\sigma_0|^2 - 2|\sigma_0|^2 \frac{\bar{\partial}\rho_0}{\rho_0} \right) d\bar{\mu}_1 \right] (4\rho_0^2)^{-1}, \end{aligned}$$

and hence we find

$$d\left(\frac{|\sigma|^2}{\kappa^2}\right) \wedge d\left(\frac{\rho + 1}{\kappa}\right) = \frac{|\mu_2|^2 |\sigma_0|^4}{2ie^{2r}\rho_0^4 |1 + \mu_1\bar{\mu}_2|^3} K d\mu_1 \wedge d\bar{\mu}_1.$$

Therefore the surface S is Weingarten iff $K = 0$.

Now suppose that $\Sigma \subset \mathbb{L}(\mathbb{H}^3)$ is of rank 1, with immersion $f : \Sigma \rightarrow \mathbb{L}(\mathbb{H}^3) : \nu = u + iv \mapsto (\mu_1(u), \mu_2(u, v))$. The induced metric $g = f^*\mathbb{G}$ has components

$$\begin{aligned} g_{uu} &= -i \left[\frac{\partial_u \mu_1 \partial_u \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} - \frac{\partial_u \bar{\mu}_1 \partial_u \mu_2}{(1 + \bar{\mu}_1 \mu_2)^2} \right] \\ g_{uv} &= g_{vu} = -\frac{i}{2} \left[\frac{\partial_u \mu_1 \partial_v \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} - \frac{\partial_u \bar{\mu}_1 \partial_v \mu_2}{(1 + \bar{\mu}_1 \mu_2)^2} \right], \end{aligned} \quad (3.7)$$

with inverse metric

$$g^{uu} = 0 \quad g^{uv} = g^{vu} = -\frac{1}{g_{uv}}. \quad (3.8)$$

The only non-vanishing Christoffel symbols are:

$$\begin{aligned} \Gamma_{uu}^u &= g^{uv} \partial_u g_{uv} - \frac{1}{2} g^{uv} \partial_v g_{uu} \\ \Gamma_{uu}^v &= g^{vv} \partial_u g_{uv} + \frac{1}{2} g^{uv} \partial_u g_{uu} - \frac{1}{2} g^{vv} \partial_v g_{uu} \\ \Gamma_{uv}^v &= \frac{1}{2} g^{uv} \partial_v g_{uu} \quad \Gamma_{vv}^v = g^{uv} \partial_v g_{uv} \end{aligned} \quad (3.9)$$

The Gauss curvature K of the surface Σ is given by

$$R_{vuvu} = K(g_{uu}g_{vv} - g_{vu}g_{vu}) = -g_{vu}^2 K. \quad (3.10)$$

But we know that

$$R_{vuvu} = g_{vi} R_{uvu}^i = g_{vu} R_{uvu}^u,$$

and after a brief computation we find that

$$R_{uvu}^u = \partial_v \Gamma_{uu}^u. \quad (3.11)$$

We first find Γ_{uu}^u . From the expressions (3.9) for the Christoffel symbols we need to find $\partial_u g_{uv}$ and $\partial_v g_{uu}$.

$$\partial_u g_{uv} = \text{Im} \left[\frac{\partial_u^2 \mu_1 \partial_v \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} + \frac{\partial_u \mu_1 \partial_u \partial_v \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} - \frac{2\mu_1 \partial_u \mu_1 \partial_u \bar{\mu}_2 \partial_v \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^3} - \frac{2\bar{\mu}_2 (\partial_u \mu_1)^2 \partial_v \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^3} \right],$$

while,

$$\frac{1}{2}\partial_v g_{uu} = \mathbb{Im} \left[\frac{\partial_u \mu_1 \partial_u \partial_v \bar{\mu}_2}{(1 + \bar{\mu}_1 \mu_2)^2} - \frac{2\mu_1 \partial_u \mu_1 \partial_u \bar{\mu}_2 \partial_v \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^3} \right].$$

Thus

$$\partial_u g_{uv} - \frac{1}{2}\partial_v g_{uu} = \mathbb{Im} \left[\frac{\partial_u^2 \mu_1 \partial_v \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} - \frac{2\bar{\mu}_2 (\partial_u \mu_1)^2 \partial_v \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^3} \right].$$

Now, the Lagrangian condition is:

$$\frac{\partial_u \mu_1 \partial_v \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} = -\frac{\partial_u \bar{\mu}_1 \partial_v \mu_2}{(1 + \bar{\mu}_1 \mu_2)^2},$$

which when applied to (3.8) gives

$$g^{uv} = \frac{(1 + \mu_1 \bar{\mu}_2)^2}{i \partial_u \mu_1 \partial_v \bar{\mu}_2} = -\frac{(1 + \bar{\mu}_1 \mu_2)^2}{i \partial_u \bar{\mu}_1 \partial_v \mu_2}.$$

Finally,

$$\begin{aligned} \Gamma_{uu}^u &= g^{uv} (\partial_u g_{uv} - \frac{1}{2} \partial_v g_{uu}) \\ &= \frac{ig^{uv}}{2} \left[-\frac{\partial_u^2 \mu_1 \partial_v \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} + \frac{\partial_u^2 \bar{\mu}_1 \partial_v \mu_2}{(1 + \bar{\mu}_1 \mu_2)^2} + \frac{2\bar{\mu}_2 (\partial_u \mu_1)^2 \partial_v \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^3} - \frac{2\mu_2 (\partial_u \bar{\mu}_1)^2 \partial_v \mu_2}{(1 + \bar{\mu}_1 \mu_2)^3} \right] \\ &= \mathbb{Im} \left[\frac{\partial_u^2 \mu_1 \partial_v \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} \frac{(1 + \mu_1 \bar{\mu}_2)^2}{i \partial_u \mu_1 \partial_v \bar{\mu}_2} + \frac{\partial_u^2 \bar{\mu}_1 \partial_v \mu_2}{(1 + \bar{\mu}_1 \mu_2)^2} \frac{(1 + \bar{\mu}_1 \mu_2)^2}{i \partial_u \bar{\mu}_1 \partial_v \mu_2} \right] \\ &= -\mathbb{Re} \left(\frac{\partial_u^2 \mu_1}{\partial_u \mu_1} - \frac{2\bar{\mu}_2 \partial_u \mu_1}{1 + \mu_1 \bar{\mu}_2} \right). \end{aligned}$$

Now if we derive the above expression with respect on v , we get:

$$\begin{aligned} \partial_v \Gamma_{uu}^u &= -\frac{1}{2} \partial_v \left(\frac{\partial_u^2 \mu_1}{\partial_u \mu_1} + \frac{\partial_u^2 \bar{\mu}_1}{\partial_u \bar{\mu}_1} - \frac{2\bar{\mu}_2 \partial_u \mu_1}{1 + \mu_1 \bar{\mu}_2} - \frac{2\mu_2 \partial_u \bar{\mu}_1}{1 + \bar{\mu}_1 \mu_2} \right) \\ &= \partial_v \left(\frac{\bar{\mu}_2 \partial_u \mu_1}{1 + \mu_1 \bar{\mu}_2} + \frac{\mu_2 \partial_u \bar{\mu}_1}{1 + \bar{\mu}_1 \mu_2} \right) \\ &= \frac{\partial_v \bar{\mu}_2 \partial_u \mu_1}{(1 + \mu_1 \bar{\mu}_2)^2} + \frac{\partial_v \mu_2 \partial_u \bar{\mu}_1}{(1 + \bar{\mu}_1 \mu_2)^2} \\ &= 0. \end{aligned}$$

From (3.11) we have that $R_{uvu}^u = \partial_v \Gamma_{uu}^u = 0$, and so $R_{vuv}^u = g_{vu} R_{uvu}^u = 0$. Therefore the Gauss curvature K given is

$$K = -\frac{R_{vuv}^u}{g_{uv}^2} = 0.$$

Thus Σ is scalar flat.

We now prove the converse: that if $\Sigma \subset \mathbb{L}(\mathbb{H}^3)$ is a Lagrangian surface of rank 1, then its orthogonal surface $S \subset \mathbb{H}^3$ is Weingarten.

Let λ_1, λ_2 be the principal curvatures of S . Then, by Proposition 7

$$\kappa = \lambda_1 \lambda_2 - 1 \quad |\sigma| = \frac{1}{2} |\lambda_1 - \lambda_2|, \quad (3.12)$$

where κ is the Gauss curvature of S and σ is the shear of Σ .

Since Σ is assumed to be of rank 1, it is given by $\mu_1 = \mu_1(u)$ and $\mu_2 = \mu_2(u, v)$. We know that the shear is

$$\sigma = \frac{8\mu_2 J_{\bar{2}\bar{1}}}{\bar{\mu}_2 \Delta |1 + \mu_1 \bar{\mu}_2|^2},$$

and therefore,

$$|\sigma|^2 = \frac{64 J_{\bar{2}\bar{1}} J_{12}}{\Delta^2 |1 + \mu_1 \bar{\mu}_2|^4} = \frac{16}{\Delta^2} \frac{\partial_u \mu_1 \partial_u \bar{\mu}_1 \partial_v \mu_2 \partial_v \bar{\mu}_2}{|1 + \mu_1 \bar{\mu}_2|^4}. \quad (3.13)$$

On the other hand, the Gauss curvature κ of S is

$$\begin{aligned} \kappa &= \frac{8}{\Delta} \left[\frac{J_{2\bar{1}}}{(1 + \bar{\mu}_1 \mu_2)^2} + \frac{J_{1\bar{2}}}{(1 + \mu_1 \bar{\mu}_2)^2} \right] \\ &= \frac{8}{\Delta} \left[-\frac{i \partial_u \bar{\mu}_1 \partial_v \mu_2}{2(1 + \bar{\mu}_1 \mu_2)^2} + \frac{i \partial_u \mu_1 \partial_v \bar{\mu}_2}{2(1 + \mu_1 \bar{\mu}_2)^2} \right] \\ &= \frac{8i}{\Delta} \frac{\partial_u \mu_1 \partial_v \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2}, \end{aligned}$$

which gives

$$\begin{aligned} \kappa^2 &= -\frac{64}{\Delta^2} \frac{\partial_u \mu_1 \partial_v \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} \frac{\partial_u \mu_1 \partial_v \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} \\ &= \frac{64}{\Delta^2} \frac{\partial_u \mu_1 \partial_v \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2} \frac{\partial_u \bar{\mu}_1 \partial_v \mu_2}{(1 + \bar{\mu}_1 \mu_2)^2} \\ &= \frac{64}{\Delta^2} \frac{\partial_u \mu_1 \partial_v \bar{\mu}_2 \partial_u \bar{\mu}_1 \partial_v \mu_2}{|1 + \mu_1 \bar{\mu}_2|^4}. \quad (3.14) \end{aligned}$$

From (3.13) and (3.14) we observe that $\kappa^2 = 4|\sigma|^2$ and substituting their expressions given in (3.12) we obtain

$$(\lambda_1 \lambda_2 - 1)^2 = (\lambda_1 - \lambda_2)^2,$$

or

$$(\lambda_1 - 1)(\lambda_1 + 1)(\lambda_2 - 1)(\lambda_2 + 1) = 0.$$

Therefore the surface S is Weingarten and this completes the rank 1 case.

A rank 0 Lagrangian surface is a horosphere, on which the metric is degenerate, and so we have completed the proof of the stated result. \square

4. EXAMPLES

4.1. Totally Null Surfaces.

Definition 8. A point γ on a surface $\Sigma \subset \mathbb{L}(\mathbb{H}^3)$ is a *totally null point* if the induced metric on Σ is identically zero at γ . A surface is *totally null* if all of its points are totally null points.

In [5] it is shown that there are two types of totally null planes: α -planes and β -planes. On the former, the anti-self-dual 2-forms vanish, while on the latter the self-dual 2-forms vanish. In particular, the α -planes are holomorphic and Lagrangian.

We now classify the totally null surfaces in $\mathbb{L}(\mathbb{H}^3)$. Before doing so, let us introduce some terminology.

Definition 9. Given a point $p \in \mathbb{H}^3$, the *geodesic sphere of radius r_0 and centre p* is the surface in \mathbb{H}^3 obtained by flowing a distance r_0 along all oriented geodesics through p .

The geodesic spheres have the following useful characterization:

Proposition 9. *The oriented normals to a geodesic sphere with centre (t_0, z_0) in the upper half space model are given by the solution set of:*

$$\bar{z}_0 \mu_1 \mu_2 + (t_0^2 + z_0 \bar{z}_0) \mu_2 - \mu_1 - z_0 = 0. \quad (4.1)$$

Proof. Let (ξ, η) be an oriented geodesic which is normal to a geodesic sphere in \mathbb{H}^3 with centre (t_0, z_0) . Then there exists $r_0 \in \mathbb{R}$ such that (see equation (1.4))

$$t_0 = \frac{|1 + \bar{\mu}_1 \mu_2|}{2|\mu_2| \cosh r_0} \quad z_0 = \frac{1 - \mu_1 \bar{\mu}_2}{2\bar{\mu}_2} + \left(\frac{1 + \mu_1 \bar{\mu}_2}{2\bar{\mu}_2} \right) \tanh r_0. \quad (4.2)$$

It is now easy to show that (4.2) implies (4.1)

Conversely, suppose (4.1) holds for an oriented geodesic $\gamma = (\mu_1, \mu_2)$. Using the coordinate change (1.3) from (μ_1, μ_2) to (ξ, η) the relation (4.1) becomes:

$$\xi(\bar{z}_0 - \bar{\eta}) - \bar{\xi}(z_0 - \eta) + |\xi|^2 \left(t_0^2 + |z_0 - \eta|^2 - \frac{1}{|\xi|^2} \right) = 0.$$

Since

$$\xi(\bar{z}_0 - \bar{\eta}) - \bar{\xi}(z_0 - \eta) \in i\mathbb{R} \quad \text{and} \quad |\xi|^2 \left(t_0^2 + |z_0 - \eta|^2 - \frac{1}{|\xi|^2} \right) \in \mathbb{R},$$

we obtain

$$\xi(\bar{z}_0 - \bar{\eta}) = \bar{\xi}(z_0 - \eta) \quad (4.3)$$

$$t_0^2 + |z_0 - \eta|^2 - \frac{1}{|\xi|^2} = 0. \quad (4.4)$$

Thus, by equation (4.3), $\bar{\xi}(z_0 - \eta)$ is real, and, by the relation (4.4) (using the fact $t_0 > 0$), we have

$$[\bar{\xi}(z_0 - \eta)]^2 < 1,$$

and therefore there is $r_0 \in \mathbb{R}$ such that

$$\bar{\xi}(z_0 - \eta) = \tanh r_0. \quad (4.5)$$

Substituting (4.5) into (4.4), and using again the fact $t_0 > 0$ we obtain

$$t_0 = \frac{1}{|\xi| \cosh r_0} \quad \text{or} \quad z_0 = \eta + \frac{\tanh r_0}{\bar{\xi}}.$$

Therefore $(t_0, z_0) \in \gamma$ (see equation (1.4)) and hence γ belongs to the sphere in $\mathbb{L}(\mathbb{H}^3)$ with centre (t_0, z_0) . \square

If we fix a point μ_0 on \mathbb{P}^1 (considered as the boundary of the ball model of \mathbb{H}^3) and look at all of the oriented geodesics that end at μ_0 , we obtain a surface in $\mathbb{L}(\mathbb{H}^3)$ that is readily found to be Lagrangian. The orthogonal surfaces in \mathbb{H}^3 are well-known:

Definition 10. A *horosphere* is a surface in \mathbb{H}^3 whose oriented normals end (or begin) at the same point on \mathbb{P}^1 .

4.2. α -surfaces. We now establish:

Theorem 2. *An immersed surface Σ in $\mathbb{L}(\mathbb{H}^3)$ is an α -surface iff Σ is the oriented normal congruence of*

- (1) *a geodesic sphere, or*
- (2) *a horosphere, or*
- (3) *a totally geodesic surface*

in \mathbb{H}^3 .

Proof. Straight-forward calculations verify that the oriented normal congruence to the three listed classes of surfaces are holomorphic and Lagrangian.

Conversely, assume that Σ is a holomorphic and Lagrangian. If it is of rank 0, then it is a horosphere.

Suppose then that the surface is of rank 1. Thus locally the surface can be parameterized by $\nu = u + iv \rightarrow (\mu_1(u), \mu_2(u, v))$. A straightforward computation shows that

$$J_{2\bar{1}} = -\frac{i}{2}\partial_u\bar{\mu}_1\partial_v\bar{\mu}_2$$

Holomorphicity implies that $J_{2\bar{1}} = 0$, and so either $\partial_u\bar{\mu}_1 = 0$ or $\partial_v\bar{\mu}_2 = 0$. Thus, either the surface is not of rank 1, or it is not immersed, respectively.

Now assume that the surface is of rank 2 and parameterize the surface Σ with μ_1 . The vanishing of the shear implies that μ_2 is a holomorphic function of μ_1 . Thus, about any given $\gamma \in \Sigma$, it can be expanded in a power series:

$$\mu_2 = \sum_{n=0}^{\infty} A_n \mu_1^n, \quad (4.6)$$

where A_n are complex numbers. The Lagrangian condition (3.2) says that

$$(1 + \mu_1\bar{\mu}_2)^2\partial\mu_2 = (1 + \bar{\mu}_1\mu_2)^2\bar{\partial}\bar{\mu}_2.$$

Inserting the series (4.6) in this, the holomorphic terms lead to the following relation:

$$A_1 - \bar{A}_1 + \sum_{n=1}^{\infty} [(n+1)A_{n+1} + 2nA_n\bar{A}_0 + (n-1)A_{n-1}\bar{A}_0^2] \mu_1^n = 0.$$

Thus A_1 is real and we get the recursion relations

$$(n+1)A_{n+1} + 2nA_n\bar{A}_0 + (n-1)A_{n-1}\bar{A}_0^2 = 0 \quad n = 1, 2, 3, \dots$$

It can be easily proven by induction that this is equivalent to

$$A_n = (-1)^{n-1}\bar{A}_0^{n-1}A_1,$$

for any $n = 1, 2, 3, \dots$, and therefore

$$\mu_2 = A_0 + \sum_{n=1}^{\infty} (-1)^{n-1}\bar{A}_0^{n-1}A_1\mu_1^n = \frac{A_0 + (A_0\bar{A}_0 + A_1)\mu_1}{1 + \bar{A}_0\mu_1}, \quad (4.7)$$

where $A_0 \in \mathbb{C}$ and $A_1 \in \mathbb{R}$.

If $A_1 > 0$, Σ is a geodesic sphere with centre (z_0, t_0) with

$$z_0 = \frac{A_0}{A_1 + A_0\bar{A}_0} \quad t_0 = \frac{\sqrt{A_1}}{A_1 + A_0\bar{A}_0},$$

as can be seen by inserting (4.7) in (4.1).

On the other hand, if $A_1 = 0$, Σ is clearly a horosphere because $\mu_2 = A_0$.

We now prove that if $A_1 < 0$ then there exists an orthogonal surface in \mathbb{H}^3 which is totally geodesic.

For the Lagrangian geodesic congruence given by (4.7), the orthogonal surfaces in \mathbb{H}^3 are obtained by integrating (2.5), which yields

$$2r = \ln |A_0 + (A_1 + A_0 \bar{A}_0) \mu_1|^2 + C, \quad (4.8)$$

where $C \in \mathbb{R}$. The divergence can now be computed by (2.4) and the result is:

$$\rho = -\frac{A_1 e^C + 1}{A_1 e^C - 1}.$$

Thus the orthogonal surface in \mathbb{H}^3 obtained by setting $C = -\ln(-A_1)$ in (4.8) is totally geodesic, since both σ and ρ vanish. \square

Note: In proving the preceding, we have established the well known classification of totally umbilic surfaces in \mathbb{H}^3 [13].

4.3. β -surfaces. In [5] the β -surfaces are classified:

Theorem 3. *A β -surface in $\mathbb{L}(\mathbb{H}^3)$ is a piece of a torus which, up to isometry, is either*

- (1) $\mathbb{L}(\mathbb{H}^2)$, where $\mathbb{H}^2 \subset \mathbb{H}^3$, or
- (2) $\mathcal{C}_1 \times \mathcal{C}_2 \subset S^2 \times S^2 - \bar{\Delta}$, where \mathcal{C}_1 is a circle given by the intersection of the 2-sphere and a plane containing the north pole, and \mathcal{C}_2 is the image of \mathcal{C}_1 under reflection in the horizontal plane through the origin.

4.4. **The Holomorphic Structure of $\mathbb{L}(\mathbb{H}^3)$.** Consider $\mathbb{P}^1 \times \mathbb{P}^1$. Then the homology group $H_2(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z} \oplus \mathbb{Z}$ has generators $[h]$ and $[v]$, where h and v are horizontal and vertical fibres of the natural projections. The intersection pairing on these generators is obviously

$$[h] \cdot [v] = 1 \quad [h] \cdot [h] = [v] \cdot [v] = 0.$$

It is also not hard to show that $[\Delta] = [h] + [v]$ and $[\bar{\Delta}] = [h] - [v]$.

Let C be a closed holomorphic curve in $\mathbb{P}^1 \times \mathbb{P}^1$. Then

$$[C] = m[h] + n[v],$$

where $m \geq 0$, $n \geq 0$ and, with m and n not both zero. Clearly

$$[C] \cdot [\Delta] = m + n > 0,$$

so every closed holomorphic curve in $\mathbb{P}^1 \times \mathbb{P}^1$ must intersect the diagonal. We conclude that there are no closed holomorphic curves in $\mathbb{P}^1 \times \mathbb{P}^1 - \Delta$. If $\mathbb{L}(\mathbb{H}^3)$ were biholomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 - \Delta$, then by Proposition 7, there would be no closed totally umbilic surfaces in \mathbb{H}^3 , which we have just shown is not true (the geodesic spheres being the counter-examples). No such restriction applies to $\mathbb{P}^1 \times \mathbb{P}^1 - \bar{\Delta}$, since there are many closed holomorphic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ that do not intersect $\bar{\Delta}$, one example being Δ :

$$[\Delta] \cdot [\bar{\Delta}] = 1 - 1 = 0.$$

Thus, the distinction between $\mathbb{P}^1 \times \mathbb{P}^1 - \Delta$ and $\mathbb{P}^1 \times \mathbb{P}^1 - \bar{\Delta}$ is essential.

4.5. Flat Surfaces.

Definition 11. A surface S in \mathbb{H}^3 is *flat* if the Gauss curvature κ of S is zero.

Theorem 4. Let S be an oriented C^2 smooth immersed surface in \mathbb{H}^3 with normal geodesic congruence $f : \Sigma \rightarrow \mathbb{L}(\mathbb{H}^3)$ parameterized by $\mu_2 = \mu_2(\mu_1, \bar{\mu}_1)$.

Then S is flat iff μ_2 is an anti-holomorphic function of μ_1 .

Proof. In this case, the Lagrangian condition for Σ is

$$\frac{\partial \mu_2}{(1 + \bar{\mu}_1 \mu_2)^2} = \frac{\bar{\partial} \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2},$$

and then, by equation (2.7), the Gauss curvature κ of S

$$\kappa = \frac{16}{\Delta} \frac{\partial \mu_2}{(1 + \bar{\mu}_1 \mu_2)^2}.$$

Therefore the vanishing of the Gauss curvature is equivalent to μ_2 being an anti-holomorphic function. We can see as well that if a Lagrangian geodesic congruence has an orthogonal flat surface in \mathbb{H}^3 then all of its orthogonal surfaces are flat. \square

4.6. Surfaces of Constant Mean Curvature 1.

Definition 12. A surface S in \mathbb{H}^3 is of *constant mean curvature 1* (CMC 1) if the mean curvature of S is equal to 1. Equivalently, the divergence is $\rho = -1$.

We now prove

Theorem 5. An immersed Lagrangian surface given by $\mu_2 = \mu_2(\mu_1, \bar{\mu}_1)$ has an orthogonal CMC 1 surface if and only if σ_0 is holomorphic, where

$$\sigma_0 = \frac{\partial \bar{\mu}_2}{(1 + \mu_1 \bar{\mu}_2)^2}.$$

Proof. Let S be an oriented C^2 smooth surface with constant mean curvature $H=1$. Let $\Sigma \subset \mathbb{L}(\mathbb{H}^3)$ be the oriented normal geodesic congruence to S .

Since Σ is Lagrangian, by Proposition 6 there exists a function $r : \Sigma \rightarrow \mathbb{R}$ satisfying (2.5).

Now, being CMC 1 implies, from $\rho = -1$ and the expression (2.4),

$$\frac{\partial \mu_2}{(1 + \bar{\mu}_1 \mu_2)^2} - \frac{|\mu_2|^2}{|1 + \bar{\mu}_1 \mu_2|^2} e^{-2r} = 0,$$

which can be integrated to

$$r = -\frac{1}{2} \ln \left(\frac{|1 + \bar{\mu}_1 \mu_2|^2}{|\mu_2|^2} \rho_0 \right),$$

where ρ_0 is defined in equation (3.5).

Inserting this in (2.5) we obtain

$$-\partial \ln \left(\frac{|1 + \bar{\mu}_1 \mu_2|^2}{|\mu_2|^2} \rho_0 \right) = \frac{\partial \mu_2}{\mu_2 (1 + \bar{\mu}_1 \mu_2)} + \frac{\partial \bar{\mu}_2}{\bar{\mu}_2 (1 + \mu_1 \bar{\mu}_2)} + \frac{\bar{\mu}_2}{1 + \mu_1 \bar{\mu}_2}. \quad (4.9)$$

On the other hand, if a given Lagrangian surface Σ satisfies the condition (4.9), then from (2.5) we have

$$2\partial r = -\partial \ln \left(\frac{|1 + \bar{\mu}_1 \mu_2|^2}{|\mu_2|^2} \rho_0 \right),$$

and hence

$$r = -\frac{1}{2} \ln \left(\frac{|1 + \bar{\mu}_1 \mu_2|^2}{|\mu_2|^2} \rho_0 \right) + c,$$

c is real constant. We thus obtain an orthogonal surface S to the geodesic congruence Σ , parameterized by r with $c = 0$. This surface is CMC 1.

Therefore a Lagrangian geodesic congruence $\mu_2 = \mu_2(\mu_1, \bar{\mu}_1)$ has an orthogonal CMC 1 surface if and only if (4.9) is satisfied. Now we prove that (4.9) is equivalent to $\bar{\partial}\sigma_0 = 0$.

The relation (4.9) can be written

$$\frac{1 + \bar{\mu}_1 \mu_2}{\mu_2} \rho_0 + \frac{1 + \mu_1 \bar{\mu}_2}{\bar{\mu}_2} \bar{\sigma}_0 + \frac{\bar{\mu}_2}{1 + \mu_1 \bar{\mu}_2} = -\frac{\bar{\mu}_1 \partial \mu_2}{1 + \bar{\mu}_1 \mu_2} - \frac{\bar{\mu}_2 + \mu_1 \partial \bar{\mu}_2}{1 + \mu_1 \bar{\mu}_2} - \frac{\partial \rho_0}{\rho_0} + \frac{\partial \mu_2}{\mu_2} + \frac{\partial \bar{\mu}_2}{\bar{\mu}_2}.$$

Using (3.6) we get

$$\begin{aligned} \frac{1 + \bar{\mu}_1 \mu_2}{\mu_2} \rho_0 + \frac{1 + \mu_1 \bar{\mu}_2}{\bar{\mu}_2} \bar{\sigma}_0 + \frac{\bar{\mu}_2}{1 + \mu_1 \bar{\mu}_2} &= -\bar{\mu}_1 (1 + \bar{\mu}_1 \mu_2) \rho_0 - \mu_1 (1 + \mu_1 \bar{\mu}_2) \bar{\sigma}_0 \\ &\quad - \frac{\bar{\mu}_2}{1 + \mu_1 \bar{\mu}_2} - \frac{1}{\rho_0} \left(\bar{\partial} \sigma_0 - \frac{2\bar{\mu}_2}{1 + \mu_1 \bar{\mu}_2} \rho_0 \right) + \frac{\partial \mu_2}{\mu_2} + \frac{\partial \bar{\mu}_2}{\bar{\mu}_2}. \end{aligned}$$

Cancelling terms using the definitions (3.3) and (3.5), we find that $\bar{\partial}\sigma_0 = 0$. \square

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